Ch. 3 Matrices

3.1 **Matrices:** A matrix is an ordered rectangular array (means arrangement) of numbers or functions. The numbers or functions are called the elements or the entries of the matrix. Matrices are denoted by capital letters.

e.g. $A = \begin{bmatrix} 2 & -4 & 1 \\ 0 & 3 & 7 \end{bmatrix}$

3.2 **Order of a Matrix:** A matrix having m rows and n columns is called a matrix of order $m \times n$ (read as m by n).

We shall use the notation A = $\left[a_{ij}\right]_{m \times n}$ to indicate a matrix of order $m \times n$ where a_{ij} represents the element in the i[th] row and j[th] column. E.g. $A = \begin{bmatrix} 2 & -4 & 1 \\ 0 & 3 & 7 \end{bmatrix}$ this matrix has 2 rows and 3 columns, its order is 2×3.

3.3 **Types of Matrices:**

a) **Column Matrix:** A matrix having only one column is known as a column matrix.

e.g. $B = \begin{bmatrix} -8 \\ 1 \\ 9 \\ 0 \end{bmatrix}$ its order is 4×1.

b) **Row Matrix:** A matrix having only one row is known as a row matrix.

e.g. $C = \begin{bmatrix} 5 & 0 & 6 \end{bmatrix}$ its order is 1×3.

c) **Square Matrix:** A matrix in which the number of rows and columns are equal is known as square matrix.

e.g. $D = \begin{bmatrix} 4 & 2 & 1 & 7 \\ -7 & 5 & -3 & 1 \\ 9 & -1 & 4 & 6 \\ 8 & 2 & 5 & -3 \end{bmatrix}$ its order is 4×4 or simply it's a matrix of order 4.

d) **Diagonal Matrix:** A square matrix is said to be a diagonal matrix if all its non diagonal elements are zero.

e. g. $E = \begin{bmatrix} 7 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ it's a matrix of order 3 and the diagonal elements are 7, - 4 and 3.

e) **Scalar Matrix:** A square matrix is said to be a scalar matrix if all its diagonal elements are equal and the non diagonal elements are zero. e. g. $F = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

f) **Identity Matrix:** A square matrix in which all the diagonal elements are equal to 1 and rest is all zero is called an identity matrix. e. g. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

g) **Zero Matrix:** A matrix is said to be zero matrix or null matrix if all its elements are zero.

e.g. $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 3.

3.4 **Equality of Matrices:** Two matrices are said to be equal if they are of the same order and corresponding elements are equal.

e. g. If $X = \begin{bmatrix} 2 & c \\ b & 4 \end{bmatrix}$; $Y = \begin{bmatrix} d & 0 \\ -1 & a \end{bmatrix}$ and if X and Y are equal then a = 4, b = – 1, c = 0 and d = 2

3.5 **Operations on Matrices:**

a) **Addition Of Matrices:** Two or more matrices can be added only if they have the same order. The sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices.

e.g. If $A = \begin{bmatrix} 2 & -4 & 1 \\ 0 & 3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & -5 \\ 6 & -2 & 1 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 2+3 & (-4)+4 & 1+(-5) \\ 0+6 & 3+(-2) & 7+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & -4 \\ 6 & 1 & 8 \end{bmatrix}$$

b) **Difference of Matrices:** Two or more matrices can be subtracted only if they have the same order. The difference of two matrices is a matrix obtained by subtracting the corresponding elements of the given matrices.

e. g. If $A = \begin{bmatrix} 2 & -4 & 1 \\ 0 & 3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & -5 \\ 6 & -2 & 1 \end{bmatrix}$ then

$$A - B = \begin{bmatrix} 2-3 & (-4)-4 & 1-(-5) \\ 0-6 & 3-(-2) & 7-1 \end{bmatrix} = \begin{bmatrix} -1 & -8 & 6 \\ -6 & 1 & 6 \end{bmatrix}$$

c) **Scalar Multiplication Of a Matrix:** If A is a matrix and k is a scalar then kA is a matrix obtained by multiplying each element of A by the scalar k.

e. g. . If $A = \begin{bmatrix} 2 & -4 & 1 \\ 0 & 3 & 7 \end{bmatrix}$ then $2A = 2\begin{bmatrix} 2 & -4 & 1 \\ 0 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 2\times 2 & -4\times 2 & 1\times 2 \\ 0\times 2 & 3\times 2 & 7\times 2 \end{bmatrix} = \begin{bmatrix} 4 & -8 & 2 \\ 0 & 6 & 14 \end{bmatrix}$

d) **Multiplication of Matrices:** Two matrices A and B can be multiplied if and only if the number of columns of A is equal to number of rows of B.

e. g. If $A = \begin{bmatrix} 2 & -4 & 1 \\ 0 & 3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 3 & -2 \\ -1 & 1 \end{bmatrix}$ A is of order 2 x 3 and B is of order 3 x 2.

we find that number of columns in A = 3 and number of rows in B = 3. Since they are equal the product AB is defined. The order of the product matrix will be 2 x 2.

$$AB = \begin{bmatrix} 2 & -4 & 1 \\ 0 & 3 & 7 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & -4 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} & \begin{bmatrix} 2 & -4 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 3 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} & \begin{bmatrix} 0 & 3 & 7 \end{bmatrix}\begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 5+(-4)\times 3+1\times(-1) & 2\times 6+(-4)\times(-2)+1\times 1 \\ 0\times 5+3\times 3+7\times(-1) & 0\times 6+3\times(-2)+7\times 1 \end{bmatrix} = \begin{bmatrix} 10+(-12)+(-1) & 12+8+1 \\ 0+9+(-7) & 0+(-6)+7 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 21 \\ 2 & 1 \end{bmatrix}$$

what about BA?

the number of columns in B = 2 and the number of rows in A = 2. Since both are equal the product BA is defined. The order of the product matrix will be 3 x 3. (Why?)

$$BA=\begin{bmatrix}5&6\\3&-2\\-1&1\end{bmatrix}\begin{bmatrix}2&-4&1\\0&3&7\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}5&6\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}&\begin{bmatrix}5&6\end{bmatrix}\begin{bmatrix}-4\\3\end{bmatrix}&\begin{bmatrix}5&6\end{bmatrix}\begin{bmatrix}1\\7\end{bmatrix}\\\begin{bmatrix}3&-2\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}&\begin{bmatrix}3&-2\end{bmatrix}\begin{bmatrix}-4\\3\end{bmatrix}&\begin{bmatrix}3&-2\end{bmatrix}\begin{bmatrix}1\\7\end{bmatrix}\\\begin{bmatrix}-1&1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}&\begin{bmatrix}-1&1\end{bmatrix}\begin{bmatrix}-4\\3\end{bmatrix}&\begin{bmatrix}-1&1\end{bmatrix}\begin{bmatrix}1\\7\end{bmatrix}\end{bmatrix}$$

$$\begin{bmatrix}5\times2+6\times0&5\times(-4)+6\times3&5\times1+6\times7\\3\times2+(-2)\times0&3\times(-4)+(-2)\times3&3\times1+(-2)\times7\\(-1)\times2+1\times0&(-1)\times(-4)+1\times3&(-1)\times1+1\times7\end{bmatrix}=\begin{bmatrix}10&-2&47\\6&-18&-11\\-2&7&6\end{bmatrix}$$

3.6 **Properties:**

a) **Properties of matrix addition:**

i) **Commutative Law:** If A and B are matrices of the same order then A + B = B + A.

ii) **Associative Law:** If A, B and C are matrices of the same order then (A + B) + C = A + (B + C).

iii) **Additive Identity:** Null matrix or zero matrix is known as the additive identity for matrix addition.
A + O = O + A = A

iv) **Additive Inverse:** If A is any matrix, then – A is said to be its additive inverse.
A + (– A) = (– A) + A = O

b) **Properties of matrix multiplication:**

i) **Associative Law:** If A, B and C are matrices then (A B) C = A (B C) whenever the product of given matrices are defined.

ii) **Distributive Law:** For three matrices A, B and C; A(B + C) = AB + AC and (A + B)C = AC + BC whenever the product of given matrices are defined.

iii) **Multiplicative Identity:** For every square matrix A, there exist an identity matrix of same order such that AI = IA = A. (Refer 3.3f above)

iv) **Note:** If A and B are matrices such that AB = O, then it is not necessary that either A = O or B = O.
For e.g. I f A $=\begin{bmatrix}0&1\\0&1\end{bmatrix}$ and B = $\begin{bmatrix}2&3\\0&0\end{bmatrix}$ we find that the product of the two matrices is a zero matrix but A and B are non – zero matrices.

3.7 **Transpose of a Matrix:** If A is a matrix of order $m\times n$, then a matrix obtained by interchanging the rows and columns of A is called the transpose of A. The transpose of A is of order $n\times m$. It is represented by A^T or A′.

e.g. If $A=\begin{bmatrix}2&-4&1\\0&3&7\end{bmatrix}$ then the transpose of A is A^T or A′ = $\begin{bmatrix}2&0\\-4&3\\1&7\end{bmatrix}$

note that the order of A is 2 x 3 while the order of its transpose is 3 x 2.

3.8 **Properties of Transpose of a Matrix:** If A, B and C are three matrices then

a) (A′)′ = A b) (kA)′ = kA′ (where k is any constant) c) (A + B)′ = A′ + B′

d) (AB)′ = B′A′ e) (ABC)′ = C′B′A′

3.9 **Symmetric and Skew Symmetric Matrices:**

a) **Symmetric Matrix:** A square matrix A is said to be symmetric if a matrix is equal to its transpose. **A′ = A.**
e.g.

b) **Skew Symmetric Matrix:** A square matrix is said to be skew symmetric matrix if its transpose is equal to its additive inverse. **A′ = – A**. (Note that all the diagonal elements of a skew symmetric matrix are zero. Can you prove it?)

3.10 **Properties of Symmetric and Skew Symmetric Matrices:**

a) **Theorem 1:** For any square matrix A with real number entries, A + A′ is a symmetric matrix and A – A′ is a skew symmetric matrix.

b) **Theorem 2:** Any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

A = $\frac{1}{2}(A + A') + \frac{1}{2}(A - A')$ where $\frac{1}{2}(A + A')$ is symmetric and $\frac{1}{2}(A - A')$ is skew symmetric.

3.11 **Elementary Operation Of a Matrix:** There are six operations (transformations) on a matrix, three of which are due to rows and three are due to columns, which are known as elementary operations or transformations.

a) *The interchange of any two rows or columns*. The interchange of i^{th} and j^{th} rows is denoted by $R_i \leftrightarrow R_j$ and interchange of i^{th} and j^{th} columns is denoted by $C_i \leftrightarrow C_j$.

b) *The multiplication of the elements of any row (or column) by a non – zero number.* The multiplication of each element of the i^{th} row(or column) by k, where $k \neq 0$ is denoted by $R_i \rightarrow kR_i$ (or $C_i \rightarrow kC_i$)

c) *The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non zero number.* The addition to the elements of i^{th} row (or column), the corresponding elements of j^{th} row (or column) multiplied by k is denoted by $R_i \rightarrow R_i + kR_j$ (or $C_i \rightarrow C_i + kC_j$).

3.12 **Invertible Matrices:** If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the inverse matrix of A and is denoted by A^{-1}. Also A is the inverse matrix of B and is denoted by B^{-1}. [**Note:** A rectangular matrix does not possess inverse matrix.]

3.13 **Inverse of a Matrix by Elementary Operations:**

a) If we wish to find A^{-1} using row operations, then we i) write **A = IA** ii) apply a sequence of row operation on A = IA till we get, I = BA. The matrix B will be the inverse of A.

b) Similarly if we wish to find A^{-1} using column operations, then we write **A = AI** and apply a sequence of column operations on A = AI till we get, I = AB.

c) **Note:** After applying elementary operations, if we obtain all zeros in one or more rows of the matrix A on LHS, then A^{-1} does not exist.