Assignment - 2
Trigonometric Functions

1. If $\sin\alpha = k\sin\beta$; prove that $\tan\left(\frac{\alpha-\beta}{2}\right) = \frac{k-1}{k+1} \times \tan\left(\frac{\alpha+\beta}{2}\right)$

Ans. $k = \frac{\sin\beta}{\sin\alpha}$

$$RHS = \frac{k-1}{k+1} \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\frac{\sin\beta}{\sin\alpha}-1}{\frac{\sin\beta}{\sin\alpha}+1} \times \tan\left(\frac{\alpha+\beta}{2}\right)$$

$$= \frac{\sin\beta - \sin\alpha}{\sin\alpha} \times \frac{\sin\alpha}{\sin\beta+\sin\alpha} \times \tan\left(\frac{\alpha+\beta}{2}\right)$$

$$= \frac{\cancel{-1}(\sin\alpha - \sin\beta)}{\cancel{-1}(\sin\alpha + \sin\beta)} \times \frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$= \frac{\cancel{2}\cancel{\cos\left(\frac{\alpha+\beta}{2}\right)}\sin\left(\frac{\alpha-\beta}{2}\right)}{\cancel{2}\cancel{\sin\left(\frac{\alpha+\beta}{2}\right)}\cos\left(\frac{\alpha-\beta}{2}\right)} \times \frac{\cancel{\sin\left(\frac{\alpha+\beta}{2}\right)}}{\cancel{\cos\left(\frac{\alpha+\beta}{2}\right)}}$$

$$= \tan\left(\frac{\alpha-\beta}{2}\right)$$

$$= LHS$$

2. If $\tan A = 5/6$ and $\tan B = 1/11$ prove that $A + B = \pi/4$

Ans.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A+B) = \frac{5/6 + 1/11}{1 - 5/6 \times 1/11}$$

$$\tan(A+B) = \frac{61/66}{61/66}$$

$$\tan(A+B) = 1$$

$$\tan(A+B) = \tan 45^\circ$$

$$\therefore A + B = \frac{\pi}{4}$$

3. If $\tan A = \frac{x}{(x-1)}$, $\tan B = \frac{1}{(2x-1)}$ prove that $A - B = \frac{\pi}{4}$

Ans. $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$$\tan(A-B) = \frac{\frac{x}{x-1} - \frac{1}{2x-1}}{1 + \frac{x}{(x-1)(2x-1)}}$$

$$\tan(A-B) = \frac{2x^2 - x - x + 1}{(2x-1)(x-1)} \times \frac{(x-1)(2x-1)}{2x^2 - x - 2x + 1 + x}$$

$$\tan(A-B) = \frac{2x^2 - 2x + 1}{2x^2 - 2x + 1}$$

$$\tan(A-B) = 1$$

$$\tan(A-B) = \tan \pi/4$$

$$\therefore \underline{A - B = \pi/4}$$

4. If $A + B = 45°$, then prove that:

(i) $(1 + \tan A)(1 + \tan B) = 2$

Ans. $\tan(A+B) = \tan 45$

$\tan A + \tan B = 1 - \tan A \tan B$

$\tan A + \tan B + \tan A \tan B = 1$

Add 1 to LHS and RHS $\Rightarrow$

$1 + \tan A + \tan B + \tan A \tan B = 1 + 1$

$1(1 + \tan A) + \tan B(1 + \tan A) = 2$

$\therefore \underline{(1 + \tan A)(1 + \tan B) = 2}$

(ii) $\cot(A-1)\cot(B-1) = 2$

Ans.

$$\cot(A+B) = \cot 45^\circ$$

$$\frac{\cot A \cot B - 1}{\cot B + \cot A} = 1$$

$$\cot A \cot B - 1 = \cot B + \cot A$$

$$\cot A \cot B - \cot B - \cot A = 1$$

Adding (1) to both LHS and RHS $\Rightarrow$

$$\cot A \cot B - \cot A + 1 - \cot B = 1 + 1$$

$$\cot A(\cot B - 1) - 1(\cot B - 1) = 2$$

$$\therefore (\cot A - 1)(\cot B - 1) = 2$$

5. If $\tan(A+B) = m$ and $\tan(A-B) = n$, show that:

(i) $\tan 2A = \dfrac{m+n}{1-mn}$

Ans

$$\text{RHS} = \frac{m+n}{1-mn} = \frac{\tan(A+B) + \tan(A-B)}{1 - \tan(A+B)\tan(A-B)}$$

$$= \frac{\dfrac{\tan A + \tan B}{1 - \tan A \tan B} + \dfrac{\tan A - \tan B}{1 + \tan A \tan B}}{1 - \tan(A+B)\tan(A-B)}$$

$$= \frac{\tan(A+B)\tan(A-B) + \tan(A+B)(\tan A-B)}{1 - \tan(A+B)\tan(A-B)}$$

$$= \tan(A+B+A-B)$$

$$= \tan 2A = \text{LHS}$$

(ii) $\tan 2B = \dfrac{m-n}{1+mn}$

Ans.

$$\text{RHS} = \frac{m-n}{1+mn} = \frac{\tan(A+B) - \tan(A-B)}{1 + \tan(A+B)\tan(A-B)}$$

$$= \tan(A+B-A+B)$$

$$= \tan 2B$$

$$= \text{LHS}$$

6. Evaluate :

(i) $\cos 70^\circ \cos 10^\circ + \sin 70^\circ \sin 10^\circ$

Ans. $\cos 70^\circ \cos 10^\circ + \sin 70^\circ \sin 10^\circ = \cos(70-10) = \cos 60^\circ = \frac{1}{2}$

(ii) $\sin(40+\theta)\cos(10^\circ+\theta) - \cos(40^\circ+\theta)\sin(10^\circ+\theta)$

Ans. $\sin(40^\circ+\theta)\cos(10^\circ+\theta) - \cos(40^\circ+\theta)\sin(10^\circ+\theta) = \sin(40^\circ+\theta-10^\circ-\theta)$

$= \sin 30^\circ = \frac{1}{2}$

7. In quadrilateral ABCD, prove that

(i) $\sin(A+B) + \sin(C+D) = 0$

Ans. $LHS = \sin(A+B) + \sin(C+D)$

$= 2\sin\left(\frac{A+B+C+D}{2}\right)\cos\left(\frac{A+B-C-D}{2}\right)$

$= 2\sin 180^\circ \cos\left(\frac{A+B-C-D}{2}\right)$

$= 2 \times 0 \times \cos\left(\frac{A+B-C-D}{2}\right)$

$= 0 = RHS$

OR

$A+B+C+D = 360^\circ$

$A+B = 360^\circ - (C+D)$

$\sin(A+B) = \sin[360^\circ - (C+D)]$

$\sin(A+B) = -\sin(C+D)$

$\Rightarrow \sin(A+B) + \sin(C+D) = 0$

(ii) $\cos A\cos B - \cos C\cos D = \sin A\sin B - \sin C\sin D$

as above.

In a quad. $A+B = 360^\circ - (C+D)$

$\Rightarrow \cos(A+B) = \cos[360^\circ - (C+D)]$

$\Rightarrow \cos(A+B) = \cos(C+D)$

$\Rightarrow \cos A\cos B - \sin A\sin B = \cos C\cos D - \sin C\sin D$

rearranging the terms, we get

$\cos A\cos B - \cos C\cos D = \sin A\sin B - \sin C\sin D$

8. Show that $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = 2\cos\theta$

$2\cos 8\theta = 2(2\cos^2 4\theta - 1)$

$= 4\cos^2 4\theta - 2$

$LHS = \sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}}$

$= \sqrt{2+\sqrt{2+\sqrt{\cancel{2}+4\cos^2 4\theta \cancel{-2}}}}$

$= \sqrt{2+\sqrt{2+2\cos 4\theta}}$

$= \sqrt{2+\sqrt{\cancel{2}+4\cos^2 2\theta \cancel{-2}}}$

$= \sqrt{2+2\cos 2\theta}$

$= \sqrt{\cancel{2}+4\cos^2\theta \cancel{-2}}$

$= \sqrt{4\cos^2\theta}$

$= 2\cos\theta = \underline{RHS}$

9. Prove that $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

Ans. $\cos 6x = \cos 2.3x$

$= 2\cos^2 3x - 1$

$= 2(4\cos^3 x - 3\cos x)^2 - 1$

$= 2(16\cos^6 x - 24\cos^4 x + 9\cos^2 x) - 1$

$\therefore \cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$

10. Prove that $\cos 4x = 1 - 8\sin^2 x \cos^2 x$

Ans. $\cos 4x = \cos 2.2x$

$= 1 - 2\sin^2 2x$

$= 1 - 2(2\sin x \cos x)^2$

$= 1 - 4\sin^2 x \cos^2 x \times 2$

$\therefore \cos 4x = 1 - 8\sin^2 x \cos^2 x$

11. Prove that $\tan 4x = \dfrac{4\tan x\,(1 - \tan^2 x)}{1 - 6\tan^2 x + \tan^4 x}$

Ans. $\tan 4x = \tan 2.2x$

$= \dfrac{2\tan 2x}{1 - \tan^2 2x}$

$= \dfrac{2 \times 2\tan x}{1 - \tan^2 x} \times \dfrac{[1 - \tan^2 x]^2}{1 - 2\tan x + \tan^4 x - 4\tan x}$

$\therefore \tan 4x = \dfrac{4\tan x\,(1 - \tan x)}{1 - 6\tan x + \tan^4 x}$

12. Prove that :

(i) $\cos\left(\dfrac{\pi}{4} + x\right) + \cos\left(\dfrac{\pi}{4} - x\right) = \sqrt{2}\cos x$

Ans. LHS $= \cos(\pi/4 + x) + \cos(\pi/4 - x)$

$= 2\cos(\pi/4)\cos x$

$= 2 \times \frac{1}{\sqrt{2}} \times \cos x$

$= \sqrt{2}\cos x = \underline{\text{RHS}}$

(ii) $\cos(3\pi/4 + x) - \cos(3\pi/4 - x) = -\sqrt{2}\sin x$

Ans. LHS $= -2\sin 3\pi/4 \sin x$

$= -2 \times \frac{1}{\sqrt{2}} \sin x$

$= -\sqrt{2}\sin x = \underline{\text{RHS}}$

13. Prove that :

(i) $\frac{\cos 7x + \cos 5x}{\sin 7x - \sin 5x} = \cot x$

Ans. LHS $= \frac{\cos 7x + \cos 5x}{\sin 7x - \sin 5x} = \frac{2\cancel{\cos 6x}\cos x}{2\sin x\cancel{\cos 6x}} = \frac{\cos x}{\sin x}$

$= \cot x = \underline{\text{RHS}}$

(ii) $\frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = -\frac{\sin 2x}{\cos 10x}$

Ans. LHS $= \frac{\cos 9x - \cos 5x}{\sin 17x - \sin 3x} = \frac{-2\cancel{\sin 7x}\sin 2x}{2\cos 10x\cancel{\sin 7x}} = -\frac{\sin 2x}{\cos 10x} = \underline{\text{RHS}}$

(iii) $\frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \tan 4x$

Ans. LHS $= \frac{\sin 5x + \sin 3x}{\cos 5x + \cos 3x} = \frac{2\sin 4x\cancel{\cos x}}{2\cos 4x\cancel{\cos x}} = \frac{\sin 4x}{\cos 4x} = \tan 4x = \underline{\text{RHS}}$

(iv) $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \tan 2x$

Ans: LHS $= \dfrac{\sin x + \sin 3x}{\cos x + \cos 3x} = \dfrac{\cancel{2}\sin 2x \cancel{\cos x}}{\cancel{2}\cos 2x \cancel{\cos x}} = \dfrac{\sin 2x}{\cos 2x} = \tan 2x = \text{RHS}$

(v) $\dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$

Ans. LHS $= \dfrac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x}$

$= \dfrac{2\cos 3x \cos x + \cos 3x}{2\sin 3x \cos x + \sin 3x}$

$= \dfrac{\cos 3x(2\cos x + 1)}{\sin 3x(2\cos x + 1)}$

$= \cot 3x = \text{RHS}$

(vi) $\dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$

Ans: LHS $= \dfrac{\sin 5x - 2\sin 3x + \sin x}{\cos 5x - \cos x}$

$= \dfrac{2\sin 3x \cos 2x - 2\sin 3x}{-2\sin 3x \sin 2x}$

$= \dfrac{-\cancel{2\sin 3x}(1 - \cos 2x)}{-\cancel{2\sin 3x}\sin 2x}$

$= \dfrac{1 - 1 + 2\sin^2 x}{2\sin x \cos x}$

$= \dfrac{\sin x}{\cos x}$

$= \tan x = \text{RHS}$

(vii) $\dfrac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x} = \tan 4x$

Ans. $LHS = \dfrac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x}$

$= \dfrac{2\sin 4x \cos 3x + 2\sin 4x \cos x}{2\cos 4x \cos 3x + 2\cos 4x \cos x}$

$= \dfrac{2\sin 4x\ (\cos 3x + \cos x)}{2\cos 4x\ (\cos 3x + \cos x)}$

$= \tan 4x = \underline{RHS}$

(viii) $\dfrac{\sin 5x + \sin 7x + \sin 9x + \sin 11x}{\cos 5x + \cos 7x + \cos 9x + \cos 11x} = \tan 8x$

Ans. $LHS = \dfrac{\sin 5x + \sin 7x + \sin 9x) + \sin 11x}{\cos 5x + (\cos 7x + \cos 7x) + \cos 11x}$

$= \dfrac{2\sin 8x \cos 3x + 2\sin 8x \cos x}{2\cos 8x \cos 3x + 2\cos 8x \cos x}$

$= \dfrac{2\sin 8x\ (\cos 3x + \cos x}{2\cos 8x\ (\cos 3x + \cos x)}$

$= \tan 8x = \underline{RHS}$

(ix) $\dfrac{\sin 3x + \sin 5x + \sin 7x + \sin 9x}{\cos 3x + \cos 5x + \cos 7x + \cos 9x} = \tan 6x$

Ans. $LHS = \dfrac{\sin 3x + (\sin 5x + \sin 7x) + \sin 9x}{\cos 3x + (\cos 5x + \cos 7x) + \cos 9x}$

$= \dfrac{2\sin 6x \cos 3x + 2\sin 6x \cos x}{2\cos 6x \cos 3x + 2\cos 6x \cos x}$

$= \dfrac{2\sin 6x\ (\cos 3x + \cos x)}{2\cos 6x\ (\cos 3x + \cos x)}$

$= \tan 6x = \underline{RHS}$

(x) $\dfrac{\sin(x+y) - 2\sin x + \sin(x-y)}{\cos(x+y) - 2\cos x + \cos(x-y)} = \tan x$

Ans. $LHS = \dfrac{\sin(x+y) - 2\sin x + \sin(x-y)}{\cos(x+y) - 2\cos x + \cos(x-y)}$

$= \dfrac{2\sin x \cos y - 2\sin x}{2\cos x \cos y - 2\cos x}$

$= \dfrac{2\sin x\,(\cos y - 1)}{2\cos x\,(\cos y - 1)}$

$= \tan x = \underline{RHS}$

(xi) $\dfrac{(\cos x - \cos 3x)(\sin 8x + \sin 2x)}{(\sin 5x - \sin x)(\cos 4x - \cos 6x)} = 1$

Ans. $LHS = \dfrac{-2\sin 2x \sin x \times 2\cos 3x \sin 5x}{2\cos 3x \sin 2x \times 2\sin 5x \sin x} = 1 = RHS$

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14 Prove that $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma) = 4\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\beta+\gamma}{2} \times \cos\left(\dfrac{\alpha+\gamma}{2}\right)$

$LHS =$

Ans. $[\cos\alpha + \cos\beta] + [\cos\gamma + \cos(\alpha+\beta+\gamma)]$

$= 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right) + 2\cos\left(\dfrac{\gamma+\alpha+\beta+\gamma}{2}\right)\cos\left(\dfrac{\gamma-\alpha-\beta-\gamma}{2}\right)$

$= 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right) + 2\cos\left(\dfrac{\alpha+\beta+2\gamma}{2}\right)\cos\left(\dfrac{-(\alpha+\beta)}{2}\right)$

$= 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right) + 2\cos\left(\dfrac{\alpha+\beta+2\gamma}{2}\right)\cos\left(\dfrac{\alpha+\beta}{2}\right)$

$= 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\left[\cos\left(\dfrac{\alpha-\beta}{2}\right) + \cos\left(\dfrac{\alpha+\beta+2\gamma}{2}\right)\right]$

$= 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\left[2\cos\left(\dfrac{\frac{\alpha-\beta}{2} + \frac{\alpha+\beta+2\gamma}{2}}{2}\right)\cos\left(\dfrac{\frac{\alpha-\beta}{2} - \frac{\alpha+\beta+2\gamma}{2}}{2}\right)\right]$

$= 2\cos\left(\dfrac{\alpha+\beta}{2}\right)\left[2\cos\left(\dfrac{\alpha+\gamma}{2}\right)\cos\left(\dfrac{\beta+\gamma}{2}\right)\right]$

$= 4\cos\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\beta+\gamma}{2}\right)\cos\left(\dfrac{\gamma+\alpha}{2}\right) = RHS$