15. Prove that :

(i) $(\cos x - \cos y)^2 + (\sin x - \sin y)^2 = 4\sin^2\left(\frac{x-y}{2}\right)$

Ans. $\text{LHS} = (\cos x - \cos y)^2 + (\sin x - \sin y)^2$

$= 4\sin^2\left(\frac{x+y}{2}\right)\sin^2\left(\frac{x-y}{2}\right) + 4\cos^2\left(\frac{x+y}{2}\right)\sin^2\left(\frac{x-y}{2}\right)$

$= 4\sin^2\left(\frac{x-y}{2}\right)\left[\sin^2\left(\frac{x+y}{2}\right) + \cos^2\left(\frac{x+y}{2}\right)\right]$

$= 4\sin^2\left(\frac{x-y}{2}\right) = \text{RHS}$

(ii) $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 4\cos^2\left(\frac{x+y}{2}\right)$

Ans. $\text{LHS} = (\cos x + \cos y)^2 + (\sin x - \sin y)^2$

$= 4\cos^2\left(\frac{x+y}{2}\right)\cos^2\left(\frac{x-y}{2}\right) + 4\cos^2\left(\frac{x+y}{2}\right)\sin^2\left(\frac{x-y}{2}\right)$

$= 4\cos^2\left(\frac{x+y}{2}\right)\left[\cos^2\left(\frac{x-y}{2}\right) + \sin^2\left(\frac{x-y}{2}\right)\right]$

$= 4\cos^2\left(\frac{x+y}{2}\right) = \text{RHS}$

16. Prove that :

(i) $\dfrac{\sin x \sin 2x + \sin 3x \sin 6x}{\sin x \cos 2x + \sin 3x \cos 6x} = \tan 5x$

Ans. $\text{LHS} = \dfrac{\sin x \sin 2x + \sin 3x \sin 6x}{\sin x \cos 2x + \sin 3x \cos 6x}$

$= \dfrac{-\frac{1}{2}\,[2\sin x \sin 2x - 2\sin 3x \sin 6x]}{\frac{1}{2}\,[2\sin x \cos 2x + 2\sin 3x \cos 6x}$

$= \dfrac{-1[(\cos 3x - \cos x) + (\cos 9x - \cancel{3x}\cos 3x)]}{1\,(\sin 3x - \sin x) + (\sin 9x - \sin 3x)}$

$= \dfrac{\cos x - \cos 3x + \cos 3x - \cos 9x}{\sin 3x - \sin 3x + \sin 9x - \sin x}$

$$= \frac{\cos x - \cos 4x}{\sin 4x - \sin x}$$

$$= \frac{-2\sin 5x \sin 4x}{2\cos 5x \sin 4x}$$

$$= \tan 5x = RHS$$

(ii) $\dfrac{\cos 8x \cos 5x - \cos 12x \cos 9x}{\sin 8x \cos 5x + \cos 12x \sin 9x} = \tan 4x$

Ans. $LHS = \dfrac{\cos 8x \cos 5x - \cos 12x \cos 9x}{\sin 8x \cos 5x + \cos 12x \sin 9x}$

$$= \frac{\cancel{1/2}[\cos 13x + \cancel{\cos 3x} - \cos 21x - \cancel{\cos 3x}]}{\cancel{1/2}[\sin 13x + \cancel{\sin 3x} + \sin 21x - \cancel{\sin 3x}]}$$

$$= \frac{\cos 13x - \cos 21x}{\sin 13x + \sin 21x}$$

$$= \frac{-2\sin \cancel{17x} - \sin 4x}{2\cancel{\sin 17x} \cos 4x}$$

$$= \frac{\sin 4x}{\cos 4x} = \tan 4x = \underline{\underline{RHS}}$$

(iii) $\dfrac{\sin 8x \cos x - \sin 6x \cos 3x}{\cos 2x \cos x - \sin 3x \sin 4x} = \tan 2x$

Ans. $LHS = \dfrac{\sin 8x \cos x - \sin 6x \cos 3x}{\cos 2x \cos x - \sin 3x \sin 4x}$

$$= \frac{1/2\,[\sin 9x + \sin 7x - \sin 9x - \sin 3x]}{1/2\,[\cos 3x + \cos x + \cos 7x - \cos x]}$$

$$= \frac{\sin 7x - \sin 3x}{\cos 7x + \cos 3x} = \frac{2\cos 5x \sin 2x}{2\cos 5x \cos 2x} = \frac{\sin 2x}{\cos 2x} = \tan 2x = \underline{\underline{RHS}}$$

(iv) $\cos 2x \cos \dfrac{x}{2} - \cos 3x \cos \dfrac{9x}{2} = \sin 5x \sin \dfrac{5x}{2}$

Ans. $LHS = \cos 2x \cos \frac{x}{2} - \cos 3x \cos \frac{9x}{2}$

$= \frac{1}{2}\left(\cos \frac{5x}{2} + \cancel{\cos \frac{3x}{2}} - \cos \frac{15x}{2} - \cos \frac{3x}{2}\right)$

$= \frac{1}{2}\left(\cos \frac{5x}{2} - \cos \frac{15x}{2}\right)$

$= \frac{-1}{2} \times 2 \sin 5x \times - \sin \frac{5x}{2}$

$= \sin 5x \sin \frac{5x}{2} = RHS$

17. If $\tan(\alpha+\theta) = n\tan(\alpha-\theta)$, show that $(n+1)\sin 2\theta = (n-1)\sin 2\alpha$

If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

$$\frac{n}{1} = \frac{\tan(\alpha+\theta)}{\tan(\alpha-\theta)}$$

using componendo & dividendo,

$$\frac{n+1}{n-1} = \frac{\tan(\alpha+\theta)+\tan(\alpha-\theta)}{\tan(\alpha+\theta)-\tan(\alpha-\theta)} = \frac{\frac{\sin(\alpha+\theta)}{\cos(\alpha+\theta)} + \frac{\sin(\alpha-\theta)}{\cos(\alpha-\theta)}}{\frac{\sin(\alpha+\theta)}{\cos(\alpha+\theta)} - \frac{\sin(\alpha-\theta)}{\cos(\alpha-\theta)}}$$

$$\Rightarrow \frac{n+1}{n-1} = \frac{\sin(\alpha+\theta)\cos(\alpha-\theta) + \cos(\alpha+\theta)\sin(\alpha-\theta)}{\sin(\alpha+\theta)\cos(\alpha-\theta) - \cos(\alpha+\theta)\sin(\alpha-\theta)}$$

$$\Rightarrow \frac{n+1}{n-1} = \frac{\sin\left[(\alpha+\cancel{\theta}) + (\alpha-\cancel{\theta})\right]}{\sin\left[(\alpha+\theta)-(\alpha-\theta)\right]} = \frac{\sin 2\alpha}{\sin 2\theta}$$

$$\Rightarrow (n+1)\sin 2\theta = (n-1)\sin 2\alpha$$

18. Prove that :

(i) $\cos^2 x + \cos^2(x+120^\circ) + \cos^2(x-120^\circ) = 3/2$

Ans

$$\cos^2 \frac{x}{2} = \frac{1+\cos x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \quad \text{——} \ (1)$$

$$LHS = \cos^2 x + \cos^2(x+120) + \cos^2(x-120^\circ)$$

$$= \frac{1+\cos 2x}{2} + \frac{1+\cos(2x+240)}{2} + \frac{1+\cos(2x-240)}{2}$$

$$= \frac{3 + \cos 2x + \cos(2x+240) + \cos(2x-240)}{2}$$

$$= \frac{1}{2}\left[3 + \cos 2x + 2\cos 2x \cos 240\right]$$

$$= \frac{1}{2}\left[3 + \cos 2x + 2\cos 2x \cos(270-240)\right]$$

$$= \frac{1}{2}\left(3 + \cos 2x - 2\cos 2x \sin 30^\circ\right)$$

$$= \frac{1}{2}\left[3 + \cos 2x - \cancel{2}\cos 2x \times \frac{1}{\cancel{2}}\right]$$

$$= \frac{1}{2}\left(3 + \cos 2x - \cos 2x\right) = \frac{3}{2} = \underline{RHS}$$

(ii) $\sin x + \sin\left(x + \frac{2\pi}{3}\right) + \sin\left(x + \frac{4\pi}{3}\right) = 0$

Ans. $LHS = \sin x + \underline{\sin(x+120) + \sin(x+240)}$

$$= \sin x + 2\sin(x+180)\cos 60^\circ$$

$$= \sin x + \sin(180+x)$$

$$= 2\sin(x+90)\cos 90^\circ = 0 = \underline{RHS}$$

19. Prove that:

(i) $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}$

Ans. $LHS = \cos 20^\circ \cos 40^\circ \underline{\cos 60^\circ} \cos 80^\circ$

$= \frac{1}{4}\left[2\cos 20^\circ \cos 40^\circ \times \cos 80^\circ\right]$

$= \frac{1}{4}\left[(\cos 60^\circ + \cos 20^\circ)\cos 80^\circ\right]$

$= \frac{1}{4}\left[\frac{\cos 80^\circ}{2} + \frac{2\cos 20^\circ \cos 80^\circ}{2}\right]$

$= \frac{1}{8}\left[\cos 80^\circ + \cos 100^\circ + \cos 60^\circ\right]$

$= \frac{1}{8}\left[\cos 80^\circ + \cos(180 - 100)^\circ + \frac{1}{2}\right]$

$= \frac{1}{8}\left[\cos 80^\circ - \cos 80^\circ + \frac{1}{2}\right]$

$= \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} = \underline{RHS}$

(ii) $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$

Ans. $LHS = \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$

$= \sqrt{3}/2 \times -1/2\left[-2\sin 20^\circ \sin 40^\circ \times \sin 80^\circ\right]$

$= -\sqrt{3}/4\left[(\cos 60^\circ - \cos 20^\circ)\sin 80^\circ\right]$

$= -\sqrt{3}/4\left[\frac{\sin 80^\circ}{2} - \frac{1}{2} \times 2\sin 80^\circ \cos 20^\circ\right]$

$= -\sqrt{3}/4 \times 1/2\left[\sin 80^\circ - \sin 100^\circ - \sin 60^\circ\right]$

$= -\sqrt{3}/8\left[\sin 80^\circ - \sin(180^\circ - 100^\circ) - \sqrt{3}/2\right]$

$= -\sqrt{3}/8\left(\sin 80^\circ - \sin 80^\circ - \sqrt{3}/2\right)$

$= \frac{-\sqrt{3}}{8} \times \frac{-\sqrt{3}}{2} = \frac{3}{16} = \underline{RHS}$

20. Prove that:

(i) $\sin 36^\circ = \sqrt{\dfrac{10-2\sqrt{5}}{4}}$

$\sin^2 36^\circ = 1 - \cos^2 36^\circ$

$= 1 - \left(\dfrac{\sqrt{5}+1}{4}\right)^2$ see part (ii)

$= 1 - \dfrac{(5+2\sqrt{5}+1)}{16}$

$= \dfrac{16-(6+2\sqrt{5})}{16}$

$= \dfrac{10-2\sqrt{5}}{16}$

$\therefore \sin 36^\circ = \sqrt{\dfrac{10-2\sqrt{5}}{16}} = \dfrac{\sqrt{10-2\sqrt{5}}}{4}$

(ii) $\cos 36^\circ = \dfrac{\sqrt{5}+1}{4}$

$A = 18^\circ \quad 2A = 36^\circ \quad 3A = 54^\circ$

$\sin 36^\circ = \sin(90 - 54^\circ)$

$\sin 36^\circ = \cos 54^\circ$

$\sin 2A = \cos 3A$

$2\sin A\cos A = 4\cos^3 A - 3\cos A$

$2\sin A = 4\cos^2 A - 3 \quad [\because \cos A \neq 0]$

$2\sin A = 4(1-\sin^2 A) - 3$

$4\sin^2 A + 2\sin A - 1 = 0$

$\sin 18^\circ = \dfrac{-2 \pm \sqrt{4+16}}{8} = \dfrac{2(-1 \pm \sqrt{5})}{8} = \dfrac{-1 \pm \sqrt{5}}{4}$

$\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$ $\left[\dfrac{-\sqrt{5}+1}{4}\right.$ rejected as $\sin x$ cannot be $\left.(-ve)\right]$

$\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 36^\circ = 1 - 2\dfrac{(\sqrt{5}-1)^2}{16}$

$\cos 36^\circ = \dfrac{16 - 2(5 - 2\sqrt{5} + 1)}{16}$

$\cos 36^\circ = \dfrac{16 - 10 + 4\sqrt{5} - 2}{16}$

$\cos 36^\circ = \dfrac{4(1+\sqrt{5})}{16}$

$\therefore \cos 36^\circ = \dfrac{\sqrt{5}+1}{4}$

(iii) $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$

$A = 18^\circ, \quad 2A = 36^\circ, \quad 3A = 54^\circ$

$\sin 36^\circ = \sin(90 - 54)$

$\sin 36 = \cos 54^\circ$

$\sin 2A = \cos 3A$

$2\sin A\cos A = 4\cos^3 A - 3\cos A$

$2\sin A = 4\cos^2 A - 3$

$2\sin A = 4 - 4\sin^2 A - 3$

$4\sin^2 A + 2\sin A - 1 = 0$

$\sin 18^\circ = \dfrac{-2 \pm \sqrt{4+16}}{8} = \dfrac{2(-1 \pm \sqrt{5})}{8} = \dfrac{-1 \pm \sqrt{5}}{4}$

$\sin 18 = \dfrac{-1+\sqrt{5}}{4}, \dfrac{-1-\sqrt{5}}{4}$ [rejected as $\sin x$ cannot be (−ve)]

$\therefore \sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$

21. Show that $\dfrac{\sin x}{\cos 3x} + \dfrac{\sin 3x}{\cos 9x} + \dfrac{\sin 9x}{\cos 27x} = \dfrac{1}{2}(\tan 27x - \tan x)$

I term $= \dfrac{\sin x}{\cos 3x}$

multiplying num. & deno. by $2\cos x$ we get

$\dfrac{2\sin x\cos x}{2\cos 3x\cos x}$

$= \dfrac{\sin 2x}{2\cos 3x\cos x}$

$= \dfrac{\sin(3x - x)}{2\cos 3x\cos x} \quad \because 2x = 3x - x$

$= \dfrac{\sin 3x\cos x - \cos 3x\sin x}{2\cos 3x\cos x}$

$$= \frac{\sin 3x \cancel{\cos x}}{2\cos 3x \cancel{\cos x}} - \frac{\cancel{\cos 3x} \sin x}{2\cancel{\cos 3x}\cos x}$$

$$= \frac{1}{2}\left[\frac{\sin 3x}{\cos 3x} - \frac{\sin x}{\cos x}\right] = \frac{1}{2}\left[\tan 3x - \tan x\right] \quad \text{--- (1)}$$

IInd term $= \dfrac{\sin 3x}{\cos 9x}$

$$= \frac{2\sin 3x \cos 3x}{2\cos 9x \cos 3x} \quad \text{(mult. num. \& deno. by } 2\cos 3x\text{)}$$

$$= \frac{\sin 6x}{2\cos 9x \cos 3x} = \frac{\sin(9x - 3x)}{2\cos 9x \cos 3x}$$

$$= \frac{\sin 9x \cos 3x - \cos 9x \sin 3x}{2\cos 9x \cos 3x}$$

$$= \frac{\sin 9x \cancel{\cos 3x}}{2\cos 9x \cancel{\cos 3x}} - \frac{\cancel{\cos 9x} \sin 3x}{2\cancel{\cos 9x}\cos 3x}$$

$$= \frac{1}{2}\left[\frac{\sin 9x}{\cos 9x} - \frac{\sin 3x}{\cos 3x}\right] = \frac{1}{2}\left[\tan 9x - \tan 3x\right] \quad \text{--- (2)}$$

III term $= \dfrac{\sin 9x}{\cos 27x} = \dfrac{2\sin 9x \cos 9x}{2\cos 27x \cos 9x}$ (mult. num. & deno. by $\cos 9x$)

$$= \frac{\sin 18x}{2\cos 27x \cos 9x} = \frac{\sin(27x - 9x)}{2\cos 27x \cos 9x}$$

$$= \frac{\sin 27x \cos 9x - \cos 27x \sin 9x}{2\cos 27x \cos 9x}$$

$$= \frac{\sin 27x \cancel{\cos 9x}}{2\cos 27x \cancel{\cos 9x}} - \frac{\cancel{\cos 27x} \sin 9x}{2\cancel{\cos 27x}\cos 9x}$$

$$= \frac{1}{2}\left[\tan 27x - \tan 9x\right] \quad \text{--- (3)}$$

adding (1), (2) & (3) we get the result.