Exercise 2.1

Question 1:

Find the principal value of $\sin^{-1}\left(-\dfrac{1}{2}\right)$

Answer

Let $\sin^{-1}\left(-\dfrac{1}{2}\right) = y.$ Then $\sin y = -\dfrac{1}{2} = -\sin\left(\dfrac{\pi}{6}\right) = \sin\left(-\dfrac{\pi}{6}\right).$

We know that the range of the principal value branch of $\sin^{-1}$ is

$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ and $\sin\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{2}.$

Therefore, the principal value of $\sin^{-1}\left(-\dfrac{1}{2}\right)$ is $-\dfrac{\pi}{6}.$

Question 2:

Find the principal value of $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

Answer

Let $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = y.$ Then, $\cos y = \dfrac{\sqrt{3}}{2} = \cos\left(\dfrac{\pi}{6}\right).$

We know that the range of the principal value branch of $\cos^{-1}$ is

$[0, \pi]$ and $\cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}.$

Therefore, the principal value of $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ is $\dfrac{\pi}{6}.$

Question 3:

Find the principal value of $\operatorname{cosec}^{-1}(2)$

Answer

Let $\csc^{-1}(2) = y$. Then, $\quad \csc y = 2 = \csc\left(\dfrac{\pi}{6}\right).$

We know that the range of the principal value branch of $\csc^{-1}$ is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}.$

Therefore, the principal value of $\csc^{-1}(2)$ is $\dfrac{\pi}{6}.$

Question 4:

Find the principal value of $\tan^{-1}\left(-\sqrt{3}\right)$

Answer

Let $\tan^{-1}\left(-\sqrt{3}\right) = y.$ Then, $\tan y = -\sqrt{3} = -\tan\dfrac{\pi}{3} = \tan\left(-\dfrac{\pi}{3}\right).$

We know that the range of the principal value branch of $\tan^{-1}$ is

$\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and $\tan\left(-\dfrac{\pi}{3}\right)$ is $-\sqrt{3}.$

Therefore, the principal value of $\tan^{-1}\left(\sqrt{3}\right)$ is $-\dfrac{\pi}{3}.$

Question 5:

Find the principal value of $\cos^{-1}\left(-\dfrac{1}{2}\right)$

Answer

Let $\cos^{-1}\left(-\dfrac{1}{2}\right) = y.$ Then, $\cos y = -\dfrac{1}{2} = -\cos\left(\dfrac{\pi}{3}\right) = \cos\left(\pi - \dfrac{\pi}{3}\right) = \cos\left(\dfrac{2\pi}{3}\right).$

We know that the range of the principal value branch of $\cos^{-1}$ is

$[0, \pi]$ and $\cos\left(\dfrac{2\pi}{3}\right) = -\dfrac{1}{2}.$

Therefore, the principal value of $\cos^{-1}\left(-\dfrac{1}{2}\right)$ is $\dfrac{2\pi}{3}.$

Question 6:

Find the principal value of $\tan^{-1}(-1)$

Answer

Let $\tan^{-1}(-1) = y$. Then,
$$\tan y = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right).$$

We know that the range of the principal value branch of $\tan^{-1}$ is

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\left(-\frac{\pi}{4}\right) = -1.$

Therefore, the principal value of $\tan^{-1}(-1)$ is $-\frac{\pi}{4}.$

Question 7:

Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Answer

Let $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = y$. Then, $\sec y = \frac{2}{\sqrt{3}} = \sec\left(\frac{\pi}{6}\right).$

We know that the range of the principal value branch of $\sec^{-1}$ is

$[0, \pi] - \left\{\frac{\pi}{2}\right\}$ and $\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}.$

Therefore, the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is $\frac{\pi}{6}.$

Question 8:

Find the principal value of $\cot^{-1}\left(\sqrt{3}\right)$

Answer

Let $\cot^{-1}\left(\sqrt{3}\right) = y$. Then, $\cot y = \sqrt{3} = \cot\left(\frac{\pi}{6}\right).$

We know that the range of the principal value branch of $\cot^{-1}$ is $(0,\pi)$ and

$$\cot\left(\frac{\pi}{6}\right) = \sqrt{3}.$$

Therefore, the principal value of $\cot^{-1}\left(\sqrt{3}\right)$ is $\frac{\pi}{6}$.

Question 9:

Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Answer

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$. Then, $\cos y = -\frac{1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$.

We know that the range of the principal value branch of $\cos^{-1}$ is $[0,\pi]$ and

$$\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$.

Question 10:

Find the principal value of $\csc^{-1}\left(-\sqrt{2}\right)$

Answer

Let $\csc^{-1}\left(-\sqrt{2}\right) = y$. Then, $\csc y = -\sqrt{2} = -\csc\left(\frac{\pi}{4}\right) = \csc\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of $\csc^{-1}$ is

$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\} \text{ and } \csc\left(-\frac{\pi}{4}\right) = -\sqrt{2}.$$

Therefore, the principal value of $\csc^{-1}\left(-\sqrt{2}\right)$ is $-\frac{\pi}{4}$.

Question 11:

Find the value of $\tan^{-1}(1) + \cos^{-1}\left(-\dfrac{1}{2}\right) + \sin^{-1}\left(-\dfrac{1}{2}\right)$

Answer

Let $\tan^{-1}(1) = x$. Then, $\tan x = 1 = \tan \dfrac{\pi}{4}$.

$\therefore \tan^{-1}(1) = \dfrac{\pi}{4}$

Let $\cos^{-1}\left(-\dfrac{1}{2}\right) = y$. Then, $\cos y = -\dfrac{1}{2} = -\cos\left(\dfrac{\pi}{3}\right) = \cos\left(\pi - \dfrac{\pi}{3}\right) = \cos\left(\dfrac{2\pi}{3}\right)$.

$\therefore \cos^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$

Let $\sin^{-1}\left(-\dfrac{1}{2}\right) = z$. Then, $\sin z = -\dfrac{1}{2} = -\sin\left(\dfrac{\pi}{6}\right) = \sin\left(-\dfrac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(-\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$

$\therefore \tan^{-1}(1) + \cos^{-1}\left(-\dfrac{1}{2}\right) + \sin^{-1}\left(-\dfrac{1}{2}\right)$

$= \dfrac{\pi}{4} + \dfrac{2\pi}{3} - \dfrac{\pi}{6}$

$= \dfrac{3\pi + 8\pi - 2\pi}{12} = \dfrac{9\pi}{12} = \dfrac{3\pi}{4}$

Question 12:

Find the value of $\cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right)$

Answer

Let $\cos^{-1}\left(\dfrac{1}{2}\right) = x$. Then, $\cos x = \dfrac{1}{2} = \cos\left(\dfrac{\pi}{3}\right)$.

$\therefore \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$

Let $\sin^{-1}\left(\dfrac{1}{2}\right) = y$. Then, $\sin y = \dfrac{1}{2} = \sin\left(\dfrac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6}$

$\therefore \cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3} + \dfrac{2\pi}{6} = \dfrac{\pi}{3} + \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

Question 13:

Find the value of if $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$ **(B)** $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$

(C) $0 < y < \pi$ **(D)** $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$

Answer

It is given that $\sin^{-1} x = y$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

Therefore, $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$.

Question 14:

Find the value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

(A) π **(B)** $-\dfrac{\pi}{3}$ **(C)** $\dfrac{\pi}{3}$ **(D)** $\dfrac{2\pi}{3}$

Answer

Let $\tan^{-1}\sqrt{3} = x$. Then, $\tan x = \sqrt{3} = \tan\dfrac{\pi}{3}$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$

Let $\sec^{-1}(-2) = y$. Then, $\sec y = -2 = -\sec\left(\dfrac{\pi}{3}\right) = \sec\left(\pi - \dfrac{\pi}{3}\right) = \sec\dfrac{2\pi}{3}$.

We know that the range of the principal value branch of $\sec^{-1}$ is $\left[0, \pi\right] - \left\{\dfrac{\pi}{2}\right\}$.

$\therefore \sec^{-1}(-2) = \dfrac{2\pi}{3}$

Hence, $\tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2) = \dfrac{\pi}{3} - \dfrac{2\pi}{3} = -\dfrac{\pi}{3}$

Question 1:

Prove $$3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right), \ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

Answer

To prove: $$3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right), \ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

Let $x = \sin\theta$. Then, $\sin^{-1} x = \theta.$

We have,

R.H.S. = $\sin^{-1}\left(3x - 4x^3\right) = \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right)$

$= \sin^{-1}\left(\sin 3\theta\right)$

$= 3\theta$

$= 3\sin^{-1} x$

= L.H.S.

Question 2:

Prove $$3\cos^{-1} x = \cos^{-1}\left(4x^3 - 3x\right), \ x \in \left[\frac{1}{2}, 1\right]$$

Answer

To prove: $$3\cos^{-1} x = \cos^{-1}\left(4x^3 - 3x\right), \ x \in \left[\frac{1}{2}, 1\right]$$

Let $x = \cos\theta$. Then, $\cos^{-1} x = \theta$.

We have,

$$\text{R.H.S.} = \cos^{-1}\left(4x^3 - 3x\right)$$
$$= \cos^{-1}\left(4\cos^3\theta - 3\cos\theta\right)$$
$$= \cos^{-1}\left(\cos 3\theta\right)$$
$$= 3\theta$$
$$= 3\cos^{-1}x$$
$$= \text{L.H.S.}$$

Question 3:

Prove $\quad \tan^{-1}\dfrac{2}{11} + \tan^{-1}\dfrac{7}{24} = \tan^{-1}\dfrac{1}{2}$

Answer

To prove: $\quad \tan^{-1}\dfrac{2}{11} + \tan^{-1}\dfrac{7}{24} = \tan^{-1}\dfrac{1}{2}$

$$\text{L.H.S.} = \tan^{-1}\dfrac{2}{11} + \tan^{-1}\dfrac{7}{24}$$

$$= \tan^{-1}\dfrac{\dfrac{2}{11} + \dfrac{7}{24}}{1 - \dfrac{2}{11}\cdot\dfrac{7}{24}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\dfrac{\dfrac{48+77}{11\times 24}}{\dfrac{11\times 24 - 14}{11\times 24}}$$

$$= \tan^{-1}\dfrac{48+77}{264-14} = \tan^{-1}\dfrac{125}{250} = \tan^{-1}\dfrac{1}{2} = \text{R.H.S.}$$

Question 4:

Prove $\quad 2\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{31}{17}$

Answer

To prove: $\quad 2\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{31}{17}$

$$\text{L.H.S.} = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{2\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1}\frac{1}{7} \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{1}{\left(\frac{3}{4}\right)} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{\left(\frac{28+3}{21}\right)}{\left(\frac{21-4}{21}\right)}$$

$$= \tan^{-1}\frac{31}{17} = \text{R.H.S.}$$

Question 5:

Write the function in the simplest form:

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}, \; x \neq 0$$

Answer

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$$\therefore \tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\dfrac{\theta}{2}}{2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x$$

Question 6:

Write the function in the simplest form:

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}}, \quad |x| > 1$$

Answer

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}}, \quad |x| > 1$$

Put $x = \operatorname{cosec}\theta \Rightarrow \theta = \operatorname{cosec}^{-1} x$

$$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}} = \tan^{-1}\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$$

$$= \tan^{-1}\left(\frac{1}{\cot\theta}\right) = \tan^{-1}\left(\tan\theta\right)$$

$$= \theta = \operatorname{cosec}^{-1} x = \frac{\pi}{2} - \sec^{-1} x \qquad \left[\operatorname{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}\right]$$

Question 7:

Write the function in the simplest form:

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),\ x<\pi$$

Answer

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right),\ x<\pi$$

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)=\tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$=\frac{x}{2}$$

Question 8:

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right),\ 0<x<\pi$$

Answer

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$$

$$=\tan^{-1}\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$=\tan^{-1}(1)-\tan^{-1}(\tan x) \qquad\qquad \left[\tan^{-1}\frac{x-y}{1-xy}=\tan^{-1}x-\tan^{-1}y\right]$$

$$=\frac{\pi}{4}-x$$

Question 9:

Write the function in the simplest form:

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}, \; |x| < a$$

Answer

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$$

$$\text{Put } x = a\sin\theta \Rightarrow \frac{x}{a} = \sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right) = \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

$$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{a}$$

Question 10:

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), \; a > 0; \; \frac{-a}{\sqrt{3}} \le x \le \frac{a}{\sqrt{3}}$$

Answer

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$$

Put $x = a\tan\theta \Rightarrow \dfrac{x}{a} = \tan\theta \Rightarrow \theta = \tan^{-1}\dfrac{x}{a}$

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = \tan^{-1}\left(\frac{3a^2 \cdot a\tan\theta - a^3\tan^3\theta}{a^3 - 3a \cdot a^2\tan^2\theta}\right)$$

$$= \tan^{-1}\left(\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a^3\tan^2\theta}\right)$$

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right)$$

$$= \tan^{-1}\left(\tan 3\theta\right)$$

$$= 3\theta$$

$$= 3\tan^{-1}\frac{x}{a}$$

Question 11:

Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\dfrac{1}{2}\right)\right]$

Answer

Let $\sin^{-1}\dfrac{1}{2} = x$. Then, $\sin x = \dfrac{1}{2} = \sin\left(\dfrac{\pi}{6}\right)$.

$\therefore \sin^{-1}\dfrac{1}{2} = \dfrac{\pi}{6}$

$\therefore \tan^{-1}\left[2\cos\left(2\sin^{-1}\dfrac{1}{2}\right)\right] = \tan^{-1}\left[2\cos\left(2 \times \dfrac{\pi}{6}\right)\right]$

$= \tan^{-1}\left[2\cos\dfrac{\pi}{3}\right] = \tan^{-1}\left[2 \times \dfrac{1}{2}\right]$

$= \tan^{-1}1 = \dfrac{\pi}{4}$

Question 12:

Find the value of $\cot\left(\tan^{-1}a+\cot^{-1}a\right)$

Answer

$$\cot\left(\tan^{-1}a+\cot^{-1}a\right)$$

$$=\cot\left(\frac{\pi}{2}\right) \qquad \left[\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right]$$

$$=0$$

Question 13:

Find the value of $\tan\dfrac{1}{2}\left[\sin^{-1}\dfrac{2x}{1+x^2}+\cos^{-1}\dfrac{1-y^2}{1+y^2}\right]$, $|x|<1$, $y>0$ and $xy<1$

Answer

Let $x = \tan\theta$. Then, $\theta = \tan^{-1}x$.

$$\therefore \sin^{-1}\frac{2x}{1+x^2}=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}\left(\sin 2\theta\right)=2\theta=2\tan^{-1}x$$

Let $y = \tan\phi$. Then, $\phi = \tan^{-1}y$.

$$\therefore \cos^{-1}\frac{1-y^2}{1+y^2}=\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)=\cos^{-1}\left(\cos 2\phi\right)=2\phi=2\tan^{-1}y$$

$$\therefore \tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$=\tan\frac{1}{2}\left[2\tan^{-1}x+2\tan^{-1}y\right]$$

$$=\tan\left[\tan^{-1}x+\tan^{-1}y\right]$$

$$=\tan\left[\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\frac{x+y}{1-xy}$$

Question 14:

$$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$$

If , then find the value of *x*.

Answer

$$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$$

$$\Rightarrow \sin\left(\sin^{-1}\frac{1}{5}\right)\cos\left(\cos^{-1}x\right)+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\left[\sin(A+B)=\sin A\cos B+\cos A\sin B\right]$$

$$\Rightarrow \frac{1}{5}\times x+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1$$

$$\Rightarrow \frac{x}{5}+\cos\left(\sin^{-1}\frac{1}{5}\right)\sin\left(\cos^{-1}x\right)=1 \qquad ...(1)$$

Now, let $\sin^{-1}\frac{1}{5}=y$.

Then, $\sin y=\frac{1}{5}\Rightarrow\cos y=\sqrt{1-\left(\frac{1}{5}\right)^2}=\frac{2\sqrt{6}}{5}\Rightarrow y=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right)$.

$$\therefore \sin^{-1}\frac{1}{5}=\cos^{-1}\left(\frac{2\sqrt{6}}{5}\right) \qquad ...(2)$$

Let $\cos^{-1}x=z$.

Then, $\cos z=x\Rightarrow\sin z=\sqrt{1-x^2}\Rightarrow z=\sin^{-1}\left(\sqrt{1-x^2}\right)$.

$$\therefore \cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\right) \qquad ...(3)$$

From (1), (2), and (3) we have:

$$\frac{x}{5}+\cos\left(\cos^{-1}\frac{2\sqrt{6}}{5}\right)\cdot\sin\left(\sin^{-1}\sqrt{1-x^2}\right)=1$$

$$\Rightarrow \frac{x}{5}+\frac{2\sqrt{6}}{5}\cdot\sqrt{1-x^2}=1$$

$$\Rightarrow x+2\sqrt{6}\sqrt{1-x^2}=5$$

$$\Rightarrow 2\sqrt{6}\sqrt{1-x^2}=5-x$$

On squaring both sides, we get:

$$(4)(6)(1-x^2) = 25 + x^2 - 10x$$

$$\Rightarrow 24 - 24x^2 = 25 + x^2 - 10x$$

$$\Rightarrow 25x^2 - 10x + 1 = 0$$

$$\Rightarrow (5x-1)^2 = 0$$

$$\Rightarrow (5x-1) = 0$$

$$\Rightarrow x = \frac{1}{5}$$

Hence, the value of x is $\frac{1}{5}$.

Question 15:

If $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of x.

Answer

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4} \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)(x+2)+(x+1)(x-2)}{(x+2)(x-2)-(x-1)(x+1)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x^2-4}{-3}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan\left[\tan^{-1}\frac{4-2x^2}{3}\right]=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{4-2x^2}{3}=1$$

$$\Rightarrow 4-2x^2=3$$

$$\Rightarrow 2x^2=4-3=1$$

$$\Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

Hence, the value of x is $\pm\dfrac{1}{\sqrt{2}}$.

Question 16:

Find the values of $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$

Answer

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

We know that $\sin^{-1}(\sin x) = x$ if $x \in \left[-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right]$, which is the principal value branch of $\sin^{-1}x$.

Here, $\dfrac{2\pi}{3} \notin \left[\dfrac{-\pi}{2},\dfrac{\pi}{2}\right]$

Now, $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$ can be written as:

$$\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right) = \sin^{-1}\left[\sin\left(\pi - \dfrac{2\pi}{3}\right)\right] = \sin^{-1}\left(\sin\dfrac{\pi}{3}\right) \text{ where } \dfrac{\pi}{3} \in \left[\dfrac{-\pi}{2},\ \dfrac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(\sin\dfrac{2\pi}{3}\right) = \sin^{-1}\left(\sin\dfrac{\pi}{3}\right) = \dfrac{\pi}{3}$$

Question 17:

Find the values of $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$

Answer

$$\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$$

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\dfrac{\pi}{2},\ \dfrac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\dfrac{3\pi}{4} \notin \left(\dfrac{-\pi}{2},\dfrac{\pi}{2}\right).$

Now, $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$ can be written as:

$$\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right) = \tan^{-1}\left[-\tan\left(\dfrac{-3\pi}{4}\right)\right] = \tan^{-1}\left[-\tan\left(\pi - \dfrac{\pi}{4}\right)\right]$$

$$= \tan^{-1}\left[-\tan\dfrac{\pi}{4}\right] = \tan^{-1}\left[\tan\left(-\dfrac{\pi}{4}\right)\right] \text{ where } -\dfrac{\pi}{4} \in \left(\dfrac{-\pi}{2},\ \dfrac{\pi}{2}\right)$$

$$\therefore \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] = \frac{-\pi}{4}$$

Question 18:

$$\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$$

Find the values of

Answer

Let $\sin^{-1}\frac{3}{5} = x$. Then, $\sin x = \frac{3}{5} \Rightarrow \cos x = \sqrt{1 - \sin^2 x} = \frac{4}{5} \Rightarrow \sec x = \frac{5}{4}$.

$$\therefore \tan x = \sqrt{\sec^2 x - 1} = \sqrt{\frac{25}{16} - 1} = \frac{3}{4}$$

$$\therefore x = \tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \qquad \ldots(i)$$

Now, $\cot^{-1}\frac{3}{2} = \tan^{-1}\frac{2}{3} \qquad \ldots(ii) \qquad \left[\tan^{-1}\frac{1}{x} = \cot^{-1}x\right]$

Hence, $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

$$= \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right) \qquad \left[\text{Using (i) and (ii)}\right]$$

$$= \tan\left(\tan^{-1}\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \cdot \frac{2}{3}}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$$

$$= \tan\left(\tan^{-1}\frac{17}{6}\right) = \frac{17}{6}$$

Question 19:

Find the values of $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$ is equal to

(A) $\dfrac{7\pi}{6}$ **(B)** $\dfrac{5\pi}{6}$ **(C)** $\dfrac{\pi}{3}$ **(D)** $\dfrac{\pi}{6}$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0,\ \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\dfrac{7\pi}{6} \notin x \in [0,\ \pi].$

Now, $\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\dfrac{7\pi}{6}\right) = \cos^{-1}\left(\cos\dfrac{-7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \dfrac{7\pi}{6}\right)\right] \quad [\cos(2\pi + x) = \cos x]$$

$$= \cos^{-1}\left[\cos\dfrac{5\pi}{6}\right] \text{ where } \dfrac{5\pi}{6} \in [0,\ \pi]$$

$$\therefore \cos^{-1}\left(\cos\dfrac{7\pi}{6}\right) = \cos^{-1}\left(\cos\dfrac{5\pi}{6}\right) = \dfrac{5\pi}{6}$$

The correct answer is B.

Question 20:

Find the values of $\sin\left(\dfrac{\pi}{3} - \sin^{-1}\left(-\dfrac{1}{2}\right)\right)$ is equal to

(A) $\dfrac{1}{2}$ **(B)** $\dfrac{1}{3}$ **(C)** $\dfrac{1}{4}$ **(D)** 1

Answer

Let $\sin^{-1}\left(\dfrac{-1}{2}\right) = x$. Then, $\sin x = \dfrac{-1}{2} = -\sin\dfrac{\pi}{6} = \sin\left(\dfrac{-\pi}{6}\right).$

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$.

$$\therefore \sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$$

$$\therefore \sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

The correct answer is D.

Miscellaneous Solutions

Question 1:

Find the value of $\cos^{-1}\left(\cos\dfrac{13\pi}{6}\right)$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\dfrac{13\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\dfrac{13\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi + \frac{\pi}{6}\right)\right] = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right], \text{ where } \frac{\pi}{6} \in [0, \pi].$$

$$\therefore \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right] = \frac{\pi}{6}$$

Question 2:

Find the value of $\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right)$

Answer

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\dfrac{7\pi}{6} \notin \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right)$ can be written as:

$$\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left[\tan\left(2\pi - \frac{5\pi}{6}\right)\right] \qquad \left[\tan(2\pi - x) = -\tan x\right]$$

$$= \tan^{-1}\left[-\tan\left(\frac{5\pi}{6}\right)\right] = \tan^{-1}\left[\tan\left(-\frac{5\pi}{6}\right)\right] = \tan^{-1}\left[\tan\left(\pi - \frac{5\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{6}\right)\right], \text{ where } \frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left(\tan\frac{\pi}{6}\right) = \frac{\pi}{6}$$

Question 3:

Prove $2\sin^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{24}{7}$

Answer

Let $\sin^{-1}\dfrac{3}{5} = x$. Then, $\sin x = \dfrac{3}{5}$.

$$\Rightarrow \cos x = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$$

$$\therefore \tan x = \frac{3}{4}$$

$$\therefore x = \tan^{-1}\frac{3}{4} \Rightarrow \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}$$

Now, we have:

$$\text{L.H.S.} = 2\sin^{-1}\frac{3}{5} = 2\tan^{-1}\frac{3}{4}$$

$$= \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right) \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{16-9}{16}}\right) = \tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)$$

$$= \tan^{-1}\frac{24}{7} = \text{R.H.S.}$$

Question 4:

Prove $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

Answer

Let $\sin^{-1}\frac{8}{17} = x$. Then, $\sin x = \frac{8}{17} \Rightarrow \cos x = \sqrt{1-\left(\frac{8}{17}\right)^2} = \sqrt{\frac{225}{289}} = \frac{15}{17}$.

$\therefore \tan x = \frac{8}{15} \Rightarrow x = \tan^{-1}\frac{8}{15}$

$\therefore \sin^{-1}\frac{8}{17} = \tan^{-1}\frac{8}{15}$ ...(1)

Now, let $\sin^{-1}\frac{3}{5} = y$. Then, $\sin y = \frac{3}{5} \Rightarrow \cos y = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

$\therefore \tan y = \frac{3}{4} \Rightarrow y = \tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4}$ ...(2)

Now, we have:

$$\text{L.H.S.} = \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4} \qquad \big[\text{Using (1) and (2)}\big]$$

$$= \tan^{-1}\frac{\dfrac{8}{15}+\dfrac{3}{4}}{1-\dfrac{8}{15}\times\dfrac{3}{4}}$$

$$= \tan^{-1}\left(\frac{32+45}{60-24}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{77}{36} = \text{R.H.S.}$$

Question 5:

Prove $\quad \cos^{-1}\dfrac{4}{5} + \cos^{-1}\dfrac{12}{13} = \cos^{-1}\dfrac{33}{65}$

Answer

Let $\cos^{-1}\dfrac{4}{5} = x$. Then, $\cos x = \dfrac{4}{5} \Rightarrow \sin x = \sqrt{1-\left(\dfrac{4}{5}\right)^2} = \dfrac{3}{5}$.

$\therefore \tan x = \dfrac{3}{4} \Rightarrow x = \tan^{-1}\dfrac{3}{4}$

$\therefore \cos^{-1}\dfrac{4}{5} = \tan^{-1}\dfrac{3}{4}$ ...(1)

Now, let $\cos^{-1}\dfrac{12}{13} = y$. Then, $\cos y = \dfrac{12}{13} \Rightarrow \sin y = \dfrac{5}{13}$.

$\therefore \tan y = \dfrac{5}{12} \Rightarrow y = \tan^{-1}\dfrac{5}{12}$

$\therefore \cos^{-1}\dfrac{12}{13} = \tan^{-1}\dfrac{5}{12}$ ...(2)

Let $\cos^{-1}\dfrac{33}{65} = z$. Then, $\cos z = \dfrac{33}{65} \Rightarrow \sin z = \dfrac{56}{65}$.

$\therefore \tan z = \dfrac{56}{33} \Rightarrow z = \tan^{-1}\dfrac{56}{33}$

$\therefore \cos^{-1}\dfrac{33}{65} = \tan^{-1}\dfrac{56}{33}$ ...(3)

Now, we will prove that:

$$\text{L.H.S.} = \cos^{-1}\dfrac{4}{5} + \cos^{-1}\dfrac{12}{13}$$

$$= \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{5}{12} \qquad \left[\text{Using (1) and (2)}\right]$$

$$= \tan^{-1}\dfrac{\dfrac{3}{4}+\dfrac{5}{12}}{1-\dfrac{3}{4}\cdot\dfrac{5}{12}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\dfrac{36+20}{48-15}$$

$$= \tan^{-1}\dfrac{56}{33}$$

$$= \tan^{-1}\dfrac{56}{33} \qquad \left[\text{by (3)}\right]$$

$$= \text{R.H.S.}$$

Question 6:

Prove $\cos^{-1}\dfrac{12}{13} + \sin^{-1}\dfrac{3}{5} = \sin^{-1}\dfrac{56}{65}$

Answer

Let $\sin^{-1}\dfrac{3}{5} = x$. Then, $\sin x = \dfrac{3}{5} \Rightarrow \cos x = \sqrt{1 - \left(\dfrac{3}{5}\right)^2} = \sqrt{\dfrac{16}{25}} = \dfrac{4}{5}$.

$\therefore \tan x = \dfrac{3}{4} \Rightarrow x = \tan^{-1}\dfrac{3}{4}$

$\therefore \sin^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{3}{4}$ ...(1)

Now, let $\cos^{-1}\dfrac{12}{13} = y$. Then, $\cos y = \dfrac{12}{13} \Rightarrow \sin y = \dfrac{5}{13}$.

$\therefore \tan y = \dfrac{5}{12} \Rightarrow y = \tan^{-1}\dfrac{5}{12}$

$\therefore \cos^{-1}\dfrac{12}{13} = \tan^{-1}\dfrac{5}{12}$ ...(2)

Let $\sin^{-1}\dfrac{56}{65} = z$. Then, $\sin z = \dfrac{56}{65} \Rightarrow \cos z = \dfrac{33}{65}$.

$\therefore \tan z = \dfrac{56}{33} \Rightarrow z = \tan^{-1}\dfrac{56}{33}$

$\therefore \sin^{-1}\dfrac{56}{65} = \tan^{-1}\dfrac{56}{33}$ ...(3)

Now, we have:

$$\text{L.H.S.} = \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{3}{4} \qquad \left[\text{Using (1) and (2)}\right]$$

$$= \tan^{-1}\frac{\dfrac{5}{12} + \dfrac{3}{4}}{1 - \dfrac{5}{12}\cdot\dfrac{3}{4}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{20+36}{48-15}$$

$$= \tan^{-1}\frac{56}{33}$$

$$= \sin^{-1}\frac{56}{65} = \text{R.H.S.} \qquad \left[\text{Using (3)}\right]$$

Question 7:

Prove $\tan^{-1}\dfrac{63}{16} = \sin^{-1}\dfrac{5}{13} + \cos^{-1}\dfrac{3}{5}$

Answer

Let $\sin^{-1}\dfrac{5}{13} = x$. Then, $\sin x = \dfrac{5}{13} \Rightarrow \cos x = \dfrac{12}{13}$.

$\therefore \tan x = \dfrac{5}{12} \Rightarrow x = \tan^{-1}\dfrac{5}{12}$

$\therefore \sin^{-1}\dfrac{5}{13} = \tan^{-1}\dfrac{5}{12} \qquad \ldots(1)$

Let $\cos^{-1}\dfrac{3}{5} = y$. Then, $\cos y = \dfrac{3}{5} \Rightarrow \sin y = \dfrac{4}{5}$.

$\therefore \tan y = \dfrac{4}{3} \Rightarrow y = \tan^{-1}\dfrac{4}{3}$

$\therefore \cos^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{4}{3} \qquad \ldots(2)$

Using (1) and (2), we have

$$\text{R.H.S.} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$$

$$= \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3}$$

$$= \tan^{-1}\left(\frac{\dfrac{5}{12}+\dfrac{4}{3}}{1-\dfrac{5}{12}\times\dfrac{4}{3}}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{15+48}{36-20}\right)$$

$$= \tan^{-1}\frac{63}{16}$$

$$= \text{L.H.S.}$$

Question 8:

Prove
$$\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$$

Answer

$$\text{L.H.S.} = \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{7+5}{35-1}\right) + \tan^{-1}\left(\frac{8+3}{24-1}\right)$$

$$= \tan^{-1}\frac{12}{34} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$

$$= \tan^{-1}\left(\frac{138+187}{391-66}\right)$$

$$= \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1}1$$

$$= \frac{\pi}{4} = \text{R.H.S.}$$

Question 9:

$$\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), \ x \in [0, 1]$$

Prove

Answer

Let $x = \tan^2\theta$. Then, $\sqrt{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$.

$$\therefore \frac{1-x}{1+x} = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$$

Now, we have:

$$\text{R.H.S.} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta) = \frac{1}{2}\times 2\theta = \theta = \tan^{-1}\sqrt{x} = \text{L.H.S.}$$

Question 10:

Prove $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\dfrac{x}{2},\ x\in\left(0,\ \dfrac{\pi}{4}\right)$

Answer

Consider $\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$=\dfrac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^{2}}{\left(\sqrt{1+\sin x}\right)^{2}-\left(\sqrt{1-\sin x}\right)^{2}}$ (by rationalizing)

$=\dfrac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x-1+\sin x}$

$=\dfrac{2\left(1+\sqrt{1-\sin^{2}x}\right)}{2\sin x}=\dfrac{1+\cos x}{\sin x}=\dfrac{2\cos^{2}\dfrac{x}{2}}{2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}$

$=\cot\dfrac{x}{2}$

$\therefore \text{L.H.S.}=\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\cot^{-1}\left(\cot\dfrac{x}{2}\right)=\dfrac{x}{2}=\text{R.H.S.}$

Question 11:

Prove $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}x,\ -\dfrac{1}{\sqrt{2}}\le x\le 1$ [**Hint:** put $x=\cos 2\theta$]

Answer

Put $x = \cos 2\theta$ so that $\theta = \dfrac{1}{2}\cos^{-1}x$. Then, we have:

$$\text{L.H.S.} = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right) = \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= \tan^{-1}1 - \tan^{-1}(\tan\theta) \qquad\qquad \left[\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y\right]$$

$$= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x = \text{R.H.S.}$$

Question 12:

Prove $\dfrac{9\pi}{8} - \dfrac{9}{4}\sin^{-1}\dfrac{1}{3} = \dfrac{9}{4}\sin^{-1}\dfrac{2\sqrt{2}}{3}$

Answer

$$\text{L.H.S.} = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$

$$= \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right)$$

$$= \frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right) \qquad . \qquad ....(1) \quad \left[\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

Now, let $\cos^{-1}\frac{1}{3} = x$. Then, $\cos x = \frac{1}{3} \Rightarrow \sin x = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$.

$$\therefore x = \sin^{-1}\frac{2\sqrt{2}}{3} \Rightarrow \cos^{-1}\frac{1}{3} = \sin^{-1}\frac{2\sqrt{2}}{3}$$

$$\therefore \text{L.H.S.} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3} = \text{R.H.S.}$$

Question 13:

Solve $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

Answer

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x) \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow \frac{2\cos x}{1-\cos^2 x} = 2\operatorname{cosec} x$$

$$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\therefore x = \frac{\pi}{4}$$

Question 14:

Solve $\tan^{-1}\dfrac{1-x}{1+x} = \dfrac{1}{2}\tan^{-1}x, (x > 0)$

Answer

$$\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x \qquad \left[\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow x=\tan\frac{\pi}{6}$$

$$\therefore x=\frac{1}{\sqrt{3}}$$

Question 15:

Solve $\sin\left(\tan^{-1}x\right),\ |x|<1$ is equal to

(A) $\dfrac{x}{\sqrt{1-x^2}}$ **(B)** $\dfrac{1}{\sqrt{1-x^2}}$ **(C)** $\dfrac{1}{\sqrt{1+x^2}}$ **(D)** $\dfrac{x}{\sqrt{1+x^2}}$

Answer

Let $\tan^{-1}x = y$. Then, $\tan y = x \Rightarrow \sin y = \dfrac{x}{\sqrt{1+x^2}}.$

$$\therefore y=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\Rightarrow \tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$\therefore \sin\left(\tan^{-1}x\right)=\sin\left(\sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)=\frac{x}{\sqrt{1+x^2}}$$

The correct answer is D.

Question 16:

Solve $\sin^{-1}\left(1-x\right)-2\sin^{-1}x=\dfrac{\pi}{2}$, then x is equal to

(A) $0, \dfrac{1}{2}$ **(B)** $1, \dfrac{1}{2}$ **(C)** 0 **(D)** $\dfrac{1}{2}$

Answer

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \dfrac{\pi}{2}$$

$$\Rightarrow -2\sin^{-1}x = \dfrac{\pi}{2} - \sin^{-1}(1-x)$$

$$\Rightarrow -2\sin^{-1}x = \cos^{-1}(1-x) \qquad \ldots(1)$$

Let $\sin^{-1}x = \theta \Rightarrow \sin\theta = x \Rightarrow \cos\theta = \sqrt{1-x^2}$.

$$\therefore \theta = \cos^{-1}\left(\sqrt{1-x^2}\right)$$

$$\therefore \sin^{-1}x = \cos^{-1}\left(\sqrt{1-x^2}\right)$$

Therefore, from equation (1), we have

$$-2\cos^{-1}\left(\sqrt{1-x^2}\right) = \cos^{-1}(1-x)$$

Put $x = \sin y$. Then, we have:

$$-2\cos^{-1}\left(\sqrt{1-\sin^2 y}\right) = \cos^{-1}(1-\sin y)$$

$$\Rightarrow -2\cos^{-1}(\cos y) = \cos^{-1}(1-\sin y)$$

$$\Rightarrow -2y = \cos^{-1}(1-\sin y)$$

$$\Rightarrow 1-\sin y = \cos(-2y) = \cos 2y$$

$$\Rightarrow 1-\sin y = 1 - 2\sin^2 y$$

$$\Rightarrow 2\sin^2 y - \sin y = 0$$

$$\Rightarrow \sin y(2\sin y - 1) = 0$$

$$\Rightarrow \sin y = 0 \text{ or } \dfrac{1}{2}$$

$$\therefore x = 0 \text{ or } x = \dfrac{1}{2}$$

But, when $x = \dfrac{1}{2}$, it can be observed that:

$$\text{L.H.S.} = \sin^{-1}\left(1 - \frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2}$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\frac{1}{2}$$

$$= -\sin^{-1}\frac{1}{2}$$

$$= -\frac{\pi}{6} \neq \frac{\pi}{2} \neq \text{R.H.S.}$$

$\therefore x = \dfrac{1}{2}$ is not the solution of the given equation.

Thus, $x = 0$.

Hence, the correct answer is **C**.

Question 17:

Solve $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\dfrac{x-y}{x+y}$ is equal to

(A) $\dfrac{\pi}{2}$ **(B).** $\dfrac{\pi}{3}$ **(C)** $\dfrac{\pi}{4}$ **(D)** $\dfrac{-3\pi}{4}$

Answer

$$\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$$

$$= \tan^{-1}\left[\frac{\dfrac{x}{y} - \dfrac{x-y}{x+y}}{1 + \left(\dfrac{x}{y}\right)\left(\dfrac{x-y}{x+y}\right)}\right]$$

$$\left[\tan^{-1} y - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}\right]$$

$$= \tan^{-1}\left[\frac{\dfrac{x(x+y) - y(x-y)}{y(x+y)}}{\dfrac{y(x+y) + x(x-y)}{y(x+y)}}\right]$$

$$= \tan^{-1}\left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy}\right)$$

$$= \tan^{-1}\left(\frac{x^2 + y^2}{x^2 + y^2}\right) = \tan^{-1} 1 = \frac{\pi}{4}$$

Hence, the correct answer is **C**.