XII Mathematics Integral Calculus

$\int k.f(x)\,dx = k\int f(x)\,dx$	$\int [f(x) \pm g(x)]dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad , \qquad n \neq -1$	$\int dx = x + C$
$\int a^x\,dx = \frac{a^x}{\log a} + C$	$\int e^x\,dx = e^x + C$
$\int \frac{dx}{x} = \log\lvert x\rvert + C$	$\int \log x dx = x\log\lvert x\rvert - x + C$ (Not necessary to memorize)
$\int \sin x dx = -\cos x + C$	$\int \cos x dx = \sin x + C$
$\int Cosec\,x.Cot x\,dx = -Cosec x + C$	$\int Sec x.Tan x\,dx = Sec x + C$
$\int Cosec^2 x\,dx = -Cot x + C$	$\int Sec^2 x\,dx = Tan x + C$
$\int Cot x\,dx = \log\lvert\sin x\rvert + C$	$\int \tan x\,dx = \log\lvert\sec x\rvert + C \quad = -\log\lvert\cos x\rvert + C$
$\int Cosec\,x\,dx = \log\lvert Cosec x - Cot x\rvert + C = \log\left\lvert\tan\frac{x}{2}\right\rvert + C$	$\int Sec x\,dx = \log\lvert\sec x + \tan x\rvert + C = \log\left\lvert\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\rvert + C$
$\int \frac{1}{\sqrt{1-x^2}}\,dx = Sin^{-1}x + C = -Cos^{-1}x + C$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = Sec^{-1}x + C = -Cosec^{-1}x + C$
$\int \frac{1}{1+x^2}\,dx = Tan^{-1}x + C$	
$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C$	$\int \frac{1}{ax+b}\,dx = \frac{\log\lvert ax+b\rvert}{a} + C$
$\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + C$	$\int A^{ax+b}\,dx = \frac{A^{ax+b}}{a.\log A} + C$
$\int Sin(ax+b)\,dx = -\frac{Cos(ax+b)}{a} + C$	$\int Cos(ax+b)\,dx = \frac{Sin(ax+b)}{a} + C$
$\int Sec^2(ax+b)\,dx = \frac{Tan(ax+b)}{a} + C$	$\int Cosec^2(ax+b)\,dx = -\frac{Cot(ax+b)}{a} + C$

$\int Sec(ax+b)Tan(ax+b)\ dx = \frac{Sec(ax+b)}{a} + C$	$\int Cosec(ax+b)\cot(ax+b)\ dx = -\frac{Cosec(ax+b)}{a} + C$
$\int \frac{1}{1+(ax+b)^2}dx = \frac{Tan^{-1}(ax+b)}{a} + C$ $= -\frac{Cot^{-1}(ax+b)}{a} + C$	$\int \frac{1}{(ax+b)\sqrt{(ax+b)^2-1}}dx = \frac{Sec(ax+b)}{a} + C$ $= -\frac{Cosec^{-1}(ax+b)}{a} + C$
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left\|\frac{x-a}{x+a}\right\| + C$	$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left\|\frac{a+x}{a-x}\right\| + C$
$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$	
$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$	$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$
$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left\|x+\sqrt{x^2-a^2}\right\| + C$	$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left\|x+\sqrt{x^2+a^2}\right\| + C$
$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left\|x+\sqrt{x^2-a^2}\right\| + C$	
$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left\|x+\sqrt{x^2+a^2}\right\| + C$	
$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$	
$\int e^x[f(x)+f'(x)]dx = e^x f(x) + C$	
Methods of Integration: **a) <u>Substitution</u>** $\int \frac{f'(x)}{f(x)}dx = \log\|f(x)\| + C$ $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C$	• In rational expressions if degree of numerator is equal or more than the degree of denominator, do polynomial division and write dividend/divisor = quotient + (remainder/divisor). • See special types in page no. 4

b) By Parts

$$\int f(x).g(x)\mathrm{dx} = f(x)\int g(x)dx - \int [f'(x)\int g(x)dx]dx + C$$

Following rule is helpful in selecting the first and second functions: **ILATE** but is not a necessary condition.

I – Inverse Trig., L – Logarithmic function, A – Algebraic, T – Trigonometric & E – Exponential

If both functions are algebraic, take that function as 1st whose derivative is simpler.

If both functions are trigonometric, take that function as 2nd whose integral is simpler.

c) Partial Fractions

S. No.	Form of rational function	Form of partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{(x-a)}+\frac{B}{(x-b)}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)}+\frac{B}{(x-b)}+\frac{C}{(x-c)}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
5.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2+bx+c cannot be factorized further	$\frac{A}{(x-a)}+\frac{Bx+C}{x^2+bx+c}$

Definite Integral as the Limit Of A Sum:

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a)+f(a+h)+\dots f(a+\overline{n-1}h)]$$

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a)+f(a+h)+\dots f(a+\overline{n-1}h)]$$ where

$$h = \frac{b-a}{n} \to 0 \text{ as } n\to\infty$$

Properties Of Definite Integral:

P₀: $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P₁: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

P₂: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

P₃: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

P₄: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

P₅: $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

P₆: $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x)=f(x)$ and 0 if $f(2a-x)=-f(x)$

P₇:

$i)$ $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function, *i.e. if* $f(-x)=f(x)$

$ii)$ $\int_{-a}^{a} f(x)\,dx = 0$, if f is an odd function, *i.e. if* $f(-x)=-f(x)$

- *Integrals reducible to the form* $\int \frac{1}{ax^2+bx+c}\,dx$

  e.g. $\int \frac{x}{x^4+x^2+1}\,dx$, $\int \frac{(2\sin 2\theta-\cos\theta)}{6-\cos^2\theta-4\sin\theta}\,d\theta$ $\int \frac{dx}{x[6(\log x)^2+7\log x+2]}$, $\int \frac{\sin 2x}{\sqrt{\cos^4 x-\sin^2 x+2}}\,dx$,

  $\int \frac{e^{-x}}{16+9e^{-2x}}\,dx$, $\int \frac{x^2}{x^6+a^6}\,dx$, $\int \frac{1}{x(x^5+1)}\,dx$, $\int \frac{1}{x(x^n+1)}\,dx$

- *Integrals of the type* $\int \frac{px+q}{ax^2+bx+c}\,dx$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

  write numerator = A(derivative of denominator) + B i.e. px + q = A(2ax + b) + B
  find values of A & B and separate into two integrals.

- *Integrals whose denominator can be rationalized.*

  $\int \frac{1}{\sqrt{3x+4}-\sqrt{3x+1}}\,dx$, $\int \frac{\sec x}{\sec x+\tan x}\,dx$, $\int \frac{1}{1-Cosx}\,dx$

- *Integrals of the type* $\int \frac{1}{a\sin^2 x+b\cos^2 x}\,dx$, $\int \frac{1}{a+b\sin^2 x}\,dx$, $\int \frac{1}{a+b\cos^2 x}\,dx$, $\int \frac{1}{(a\sin x+b\cos x)^2}\,dx$

  $\int \frac{1}{a+b\sin^2 x+c\cos^2 x}\,dx$

  i) divide numerator and denominator both by $\cos^2 x$

  ii) replace, $\sec^2 x$, *if any in denominator by 1 +* $\tan^2 x$

  iii) put tanx = t, it implies $\sec^2 x\,dx = dt$ *and reduce the integral to the form* $\int \frac{1}{at^2+bt+c}\,dt$

- *Integrals of the type* $\int \frac{1}{a\sin x + b\cos x + c}dx$, $\int \frac{1}{a\sin x + b\cos x}dx$, $\int \frac{1}{a + b\sin x}dx$, $\int \frac{1}{a + b\cos x}dx$

  i) put sinx = $\frac{2\tan(x/2)}{1+\tan^2(x/2)}$, *cos x =* $\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$

  ii) replace $1+\tan^2(x/2)$ *in numerator by* $\sec^2(x/2)$

  iii) put tan(x/2) = t, it implies $\frac{1}{2}\sec^2 xdx = dt$ *and reduce the integral to the form* $\int \frac{1}{at^2 + bt + c}dt$

- *Integrals of the type* $\int \frac{a\sin x + b\cos x}{c\sin x + d\cos x}dx$

  write numerator = A(derivative of denominator) + B(denominator)
  find values of A and B & separate into two integrals.

- *Integrals of the type* $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}dx$

  write numerator = A(derivative of denominator) + B(denominator) + C
  find values of A, B & C, separate into three integrals.

- *Integrals of the type* $\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + C$

- *Integrals of the type* $\int (px+q)\sqrt{ax^2+bx+c}\,dx$, $\int linear\sqrt{quadratic}\,dx$

  let px + q = A(derivative of denominator) + B = A(2ax + b) + B
  find values of A and B & separate into two integrals.

- *Integrals of the type* $\int \frac{x^2+1}{x^4+\lambda x^2+1}dx$, $\int \frac{x^2-1}{x^4+\lambda x^2+1}dx$, $\int \frac{1}{x^4+\lambda x^2+1}dx$, $\int \frac{x^2+1}{x^4+1}dx$, $\int \frac{1}{x^4+1}dx$

  i) divide numerator and denominator by x^2

  ii) express denominator in the form $\left(x+\frac{1}{x}\right)^2 \pm k^2$

  iii) introduce derivative of $x+\frac{1}{x}$ *or* $x-\frac{1}{x}$ *or both in the numerator*

  iv) substitute $x+\frac{1}{x} = t$ *or* $x-\frac{1}{x} = t$ *& reduce the integral to* $\int \frac{1}{x^2 \pm a^2}dx$

<u>Integration Of some Special Irrational Algebraic Functions (optional)</u>

- *Integrals of the type* $\int \frac{1}{(px+q)\sqrt{ax^2+bx+c}}dx$, $\int \frac{1}{linear\sqrt{quadratic}}dx$

  put px + q = 1/t which implies dx = $\frac{-1}{pt^2}dt$ *and x =* $\frac{\frac{1}{t}-q}{p}$

- *Integrals of the type* $\int \frac{1}{(ax^2+bx+c)\sqrt{px+q}}dx$, $\int \frac{1}{quadratic\sqrt{linear}}dx$

  put px + q = t^2 *which implies dx =* $\frac{2tdt}{p}$ *and x =* $\frac{t^2-q}{p}$

- Integrals of the type $\int \frac{1}{(ax+b)\sqrt{px+q}}dx$, $\int \frac{1}{linear\sqrt{linear}}dx$

  put $px + q = t^2$ which implies $dx = \frac{2tdt}{p}$ and $x = \frac{t^2 - q}{p}$

- Integrals of the type $\int \frac{1}{(ax^2+b)\sqrt{px^2+q}}dx$,

  put $x = 1/t$ which implies $dx = \frac{-1}{t^2}dt$ and simplify it to obtain $\int \frac{-tdt}{(a+bt^2)\sqrt{p+qt^2}}$ in which put $p + qt^2 = u^2$