Ch. 2. Inverse Trigonometric Functions

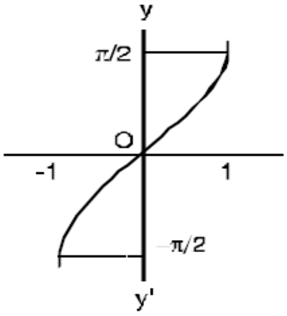
1. The domain and range of the trigonometric functions are as follows:

FUNCTION	DOMAIN	RANGE
$y = \sin x$	R	$[-1,1]$
$y = \cos x$	R	$[-1,1]$
$y = \tan x$	$R - \{n\pi : n \in I\}$	R
$y = \cos ecx$	$R - \{n\pi : n \in I\}$	$R-(-1,1)$ or $(-\infty,1] \cup [1,\infty)$
$y = \sec x$	$R - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	$R-(-1,1)$ or $(-\infty,1] \cup [1,\infty)$
$y = \cot x$	$R - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$	R

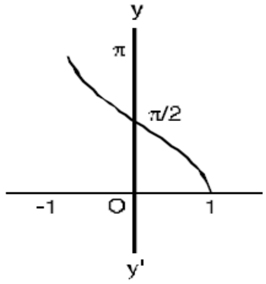
2. The domain and range of the inverse trigonometric functions are as follows:

		DOMAIN		RANGE
$\sin^{-1}$	:	$[-1, 1]$	$\rightarrow$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}$	:	$[-1, 1]$	$\rightarrow$	$[0, \pi]$
cosec^{-1}	:	$\mathbf{R} - (-1,1)$	$\rightarrow$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1}$	:	$\mathbf{R} - (-1, 1)$	$\rightarrow$	$[0, \pi] - \{\frac{\pi}{2}\}$
$\tan^{-1}$	:	$\mathbf{R}$	$\rightarrow$	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1}$	:	$\mathbf{R}$	$\rightarrow$	$(0, \pi)$

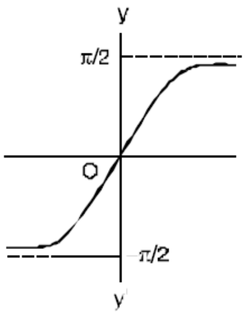
3. Graphs of Inverse Trigonometric Functions (Principal Branch)



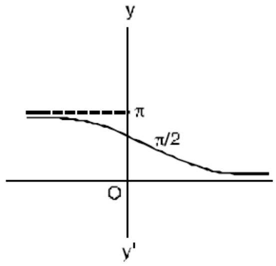
$y = \sin^{-1} x$



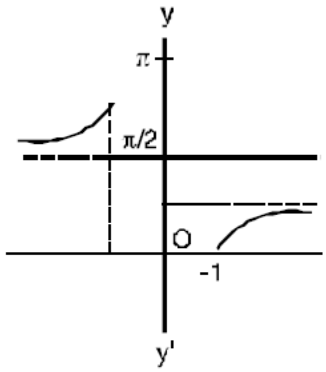
$y = \cos^{-1} x$



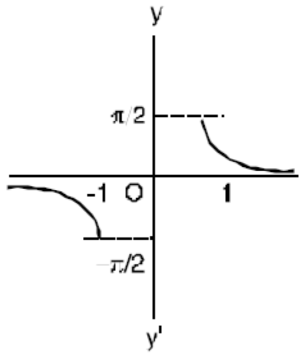
$y = \tan^{-1} x$



$y = \cot^{-1} x$



$y = \sec^{-1} x$



$y = \text{cosec}^{-1} x$

IMPORTANT FORMULAE FOR INVERSE TRIGONOMETRIC FUNCTIONS

4. $\sin^{-1}(\sin x) = x,$ $\quad$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
5. $\cos^{-1}(\cos x) = x,$ $\quad$ for all $x \in [0,\ \pi\,]$
6. $\tan^{-1}(\tan x) = x,$ $\quad$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
7. $\text{cosec}^{-1}(\text{cosec}\,x) = x,$ $\quad$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], x \neq 0$
8. $\sec^{-1}(\sec x) = x,$ $\quad$ for all $x \in [0,\ \pi\,], x \neq \frac{\pi}{2}$
9. $\cot^{-1}(\cot x) = x,$ $\quad$ for all $x \in (0,\ \pi)$

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10. $\sin(\sin^{-1}x) = x,$ $\quad$ for all $x \in [-1,\ 1\,]$
11. $\cos(\cos^{-1}x) = x,$ $\quad$ for all $x \in [-1,\ 1\,]$
12. $\tan(\tan^{-1}x) = x,$ $\quad$ for all $x \in R$
13. $\text{cosec}(\text{cosec}^{-1}x) = x,$ $\quad$ for all $x \in (-\infty, -1] \cup [1, \infty)$
14. $\sec(\sec^{-1}x) = x,$ $\quad$ for all $x \in (-\infty, -1] \cup [1, \infty)$
15. $\cot(\cot^{-1}x) = x,$ $\quad$ for all $x \in R$

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16. $\sin^{-1}(-x) = -\sin^{-1}x,$ $\quad$ for all $x \in [-1,\ 1]$
17. $\cos^{-1}(-x) = \pi - \cos^{-1}x,$ $\quad$ for all $x \in [-1,\ 1]$
18. $\tan^{-1}(-x) = -\tan^{-1}x,$ $\quad$ for all $x \in R$
19. $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x,$ for all $x \in (-\infty, -1] \cup [1, \infty)$
20. $\sec^{-1}(-x) = \pi - \sec^{-1}x,$ $\quad$ for all $x \in (-\infty, -1] \cup [1, \infty)$
21. $\cot^{-1}(-x) = \pi - \cot^{-1}x,$ $\quad$ for all $x \in R$

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22. $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x, \quad for\ all\ x \in (-\infty, -1] \cup [1, \infty)$

23. $\sin^{-1}\left(\frac{1}{x}\right) = cosec^{-1}x, \quad for\ all\ x \in (-\infty, -1] \cup [1, \infty)$

24. $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, & for\ x > 0 \\ -\pi + \cot^{-1}x, & for\ x < 0 \end{cases}$

25. $\cot^{-1}\left(\frac{1}{x}\right) = \begin{cases} \tan^{-1}x, & for\ x > 0 \\ \pi + \tan^{-1}x, & for\ x < 0 \end{cases}$

26. $cosec^{-1}\left(\frac{1}{x}\right) = \sin^{-1}x, \quad for\ all\ x \in [-1,\ 1]$

27. $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}x, \quad for\ all\ x \in [-1,\ 1]$

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28. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \quad for\ all\ x \in [-1,\ 1]$

29. $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \quad for\ all\ x \in R$

30. $\sec^{-1}x + cosec^{-1}x = \frac{\pi}{2}, \quad for\ all\ x \in (-\infty, -1] \cup [1, \infty)$

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31. $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), & if\ xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & if\ x > 0,\ y > 0\ and\ xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), & if\ x < 0,\ y < 0\ and\ xy > 1 \end{cases}$

32. $\tan^{-1}x - \tan^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right), & if\ xy > -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & if\ x > 0,\ y < 0\ and\ xy < -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & if\ x < 0,\ y > 0\ and\ xy < -1 \end{cases}$

33. $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

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34. $\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right], & if\ -1 \le x, y \le 1\ and\ x^2+y^2 \le 1 \\ \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right], & if\ 0 \le x, y \le 1\ and\ x^2+y^2 > 1 \\ -\pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right], & if\ -1 \le x, y \le 0\ and\ x^2+y^2 > 1 \end{cases}$

35. $\sin^{-1}x - \sin^{-1}y =$

$$\begin{cases} \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right], & if\ -1 \le x, y \le 1\ and\ x^2+y^2 \le 1 \\ \pi - \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right], & if\ 0 < x \le 1; -1 \le y \le 0\ and\ x^2+y^2 > 1 \\ -\pi - \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right], & if\ -1 \le x < 0; 0 < y \le 1\ and\ x^2+y^2 > 1 \end{cases}$$

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36. $\cos^{-1}x + \cos^{-1}y = \begin{cases} \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right], & if\ -1 \le x, y \le 1\ and\ x+y \ge 0 \\ 2\pi - \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right], & if\ -1 \le x, y \le 1\ and\ x+y \le 0 \end{cases}$

37. $cos^{-1}x - cos^{-1}y = \begin{cases} cos^{-1}\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right], & if\ -1 \le x, y \le 1\ and\ x - y \le 0 \\ -cos^{-1}\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right], & if\ -1 \le y \le 0;\ 0 < x \le 1\ and\ x - y \ge 0 \end{cases}$

38.

$$\sin^{-1}(x) = \cos^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

$$\cos^{-1}(x) = \sin^{-1}\left(\sqrt{1-x^2}\right) = \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$

39.

$$2\sin^{-1}(x) = \sin^{-1}\left(2x\sqrt{1-x^2}\right) = \cos^{-1}\left(1-2x^2\right),\ -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$$

$$2\cos^{-1}(x) = \sin^{-1}\left(2x\sqrt{1-x^2}\right) = \cos^{-1}\left(2x^2-1\right),\ \frac{1}{\sqrt{2}} \le x \le 1$$

$$2\tan^{-1}(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right),\ |\text{x}| \le 1$$

$$= \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),\ x \ge 0$$

$$= \tan^{-1}\left(\frac{2x}{1-x^2}\right),\ -1 < x < 1$$

40.

$$3\sin^{-1}(x) = \sin^{-1}\left(3x-4x^3\right)$$

$$3cox^{-1}(x) = \cos^{-1}\left(4x^3-3x\right)$$

$$3\tan^{-1}(x) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

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41. List of substitutions:

S. No.	Form	substitution
1	$\sqrt{x^2 - a^2}$	x = acosecθ or x = asecθ
2	$\sqrt{x^2 + a^2}$	x = atanθ or x = acotθ
3	$\sqrt{a^2 - x^2}$	x = acosθ or x = asinθ
4	$\sqrt{a-x}$ & $\sqrt{a+x}$	x = acos2θ or x = acosθ

Trigonometric Formulae

I Sum and Difference of angle formulae:

a) $\text{Sin}(x + y) = \sin x \cos y + \cos x \sin y$

b) $\text{Sin}(x - y) = \sin x \cos y - \cos x \sin y$

c) $\text{Cos}(x + y) = \cos x \cos y - \sin x \sin y$

d) $\text{Cos}(x - y) = \cos x \cos y + \sin x \sin y$

e) $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$

f) $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

g) $\cot(x + y) = \dfrac{\cot x \cot y - 1}{\cot y + \cot x}$

h) $\cot(x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$

II Compound angle Formulae

a) $\sin 2x = 2\sin x \cos x = \dfrac{2\tan x}{1 + \tan^2 x}$

b) $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1 = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

c) $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

d) $1 - \cos 2x = 2\sin^2 x$

e) $1 + \text{Cos } 2x = 2\cos^2 x$

f) $1 - \cos x = 2\sin^2 \dfrac{x}{2}$

g) $1 + \cos x = 2\cos^2 \dfrac{x}{2}$

h) $\text{Sin}x = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2} = \dfrac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$

i) $\text{Cos}x = 2\cos^2\left(\dfrac{x}{2}\right) - 1 = 1 - 2\sin^2\left(\dfrac{x}{2}\right) = \cos^2\left(\dfrac{x}{2}\right) - \sin^2\left(\dfrac{x}{2}\right) = \dfrac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}$

j) $\text{Sin } 3x = 3\sin x - 4\sin^3 x$

k) $\text{Cos } 3x = 4\cos^3 x - 3\cos x$

l) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

III **Transformation Formulae (converting product of trig. Functions into sum/difference)**

a) 2 sin x cosy = sin (x + y) + sin (x – y)

b) 2 cosx sin y = sin(x + y) – sin (x – y)

c) 2 cosx cosy = cos(x + y) + cos (x – y)

d) – 2 sin x siny = cos(x + y) – cos (x – y) or cos(x – y) – cos (x + y) = 2 sin x siny

IV. Transformation Formulae (Converting sum/difference of trig. Functions into product)

a) sin x + sin y = $2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

b) sin x − sin y = $2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$

c) cos x + cos y = $2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

d) cos x − cos y = $-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

V. Solution of Trigonometric Equations:

1. a) sinx = 0 x = $n\pi;\ n\in Z$

   b) cos x = 0 x = $(2n+1)\frac{\pi}{2};\ n\in Z$

   c) tan x = 0 x = $n\pi;\ n\in Z$

2. a) sin x = sin y x = $n\pi+(-1)^n y;\ n\in Z$

   b) cos x = cos y x = $2n\pi\pm y;\ n\in Z$

   c) tan x = tan y x = $n\pi+y;\ n\in Z$

VI Sine and Cosine rules (Not required for exam):

a) Sine rule: If a, b, c are sides opposite to angles A, B and C of ΔABC respectively, then

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

b) Cosine rule:

1. $a^2=b^2+c^2-2bc\cos A$ **OR** cosA = $\frac{b^2+c^2-a^2}{2bc}$

2. $b^2=c^2+a^2-2ca\cos B$ **OR** cosB = $\frac{c^2+a^2-b^2}{2ca}$

3. $c^2=a^2+b^2-2ab\cos C$ **OR** cosC = $\frac{a^2+b^2-c^2}{2ab}$